Single Parameter Risk Measures and Multiple Sources of Risk: A Re-Examination of the Data Based on Changes in Determinants of Price Over Time

by

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I. Introduction

The debut of the concept of a market equilibrium of expected returns determined by the covariance of the assets returns with those of a well diversified portfolio has provided the impetus for a considerable amount of research.[1] This work has been divided between applications of the model results and empirical investigations of the nature of risk in order to test the conformance of observed data to the model. While the applications have been interesting and potentially useful, the applicability of the Capital Asset Pricing Model[2] to actual markets for risky assets has yet to be shown. There are two possible explanations for this failure: either the model is inappropriate for real markets, or the tests which have been applied have required assumptions which are not met by the actual data. Although the former case may eventually be confirmed, this would require a particular perversity on the part of the actual markets in view of the robustness of the model to possible failures in the underlining assumptions. For example Stone has shown that the usual market equilibrium relationship is still a good first approximation to the personal equilibrium condition, even when probability distributions and utility functions are generalized when no particular agreement exists, and even without the necessity of a riskless asset. Because of this it still seems reasonable to look for ways in which the failure of tests may be explained by the assumptions which relate the tests to the model being tested, or by econometric problems[3] rather than to abandon the capital asset pricing model and look for new equilibrium conditions.

The first problem facing anyone wishing to test the capital asset pricing model is to find some observable data which can be used to represent the variables which comprise the model. It is at this point that the research described in this paper takes issue with all previous efforts to test the capital asset pricing model relationship. The market equilibrium relationship can be written

$$\bar{r}_{it} = r_{ft} + \beta_{it}(\bar{r}_{mt} - r_{ft}) \qquad (1\text{-}1)$$

$\bar{r}_{it}$ = expected return of asset i as of time t for one decision interval

r_{ft} = risk free rate of return as of time t for one decision interval

$\bar{r}_{mt}$ = same as $\bar{r}_{it}$ where m is the portfolio made up of value weighted proportions of all risky assets

$\beta_{it} = \dfrac{Cov_t(r_{it}, r_{mt})}{Var_t(r_{mt})}$ = the anticipated covariance at time t of returns to asset it with returns to the market portfolio divided by variance at time t of market returns, both taken as anticipated for a single decision period.

Thus far no one has been able to observe expected returns for assets or the market portfolio nor has anyone been able to gather anticipations data about the values of β_i. Inference about the model has, therefore, been gained by using what appear historically to be reasonable forecasts for the β_i values, and assuming that by averaging the subsequent return data that the average realized return data should be an unbiased indicator of previously anticipated means over the averaging period. It will be shown in the next two sections that neither of these approaches is as reasonable or secure as it first appears and that they should not be assumed to hold without considerable further justification.

II. Returns and Equilibria

In order to examine the relationship of realized returns to expected returns more closely, it is convenient to consider explicitly the manner in which returns are produced as shown in equation (2-1).

$$\text{(2-1)} \qquad \tilde{r}_{it} = \frac{\tilde{p}_{i\,t+1} + \tilde{c}_{i\,t+1} - p_{it}}{p_{it}}$$

p_{it} = Price of asset at time t (known);

$\tilde{p}_{i,t+1}$ = Price of asset it at time t+1 (random variable);

$\bar{p}_{i,t+1}$ = Expected price of asset i at time t+1;

$\tilde{c}_{i,t+1}$ = Cash throw off generated by the asset between t and t+1 payment is assumed to be at end of period);

$\tilde{r}_{it}$ = Actual return to asset i between t and t+1;

$\bar{r}_{it}$ = Expected return to asset i between t and t+1.

Returns are seen to result from cash throw offs produced over the period and the price of the asset at two different times. Since price is the variable which adjusts to an equilibrium level, the realized return is actually dependent on equilibrium conditions at two different times; t and t+1. For p_t to adjust to an equilibrium level of expected return at t it is only necessary to know the expected equilibrium level of price at t+1 since

$$\text{(2-2)} \quad \bar{r}_{it} = \frac{E_t[\tilde{p}_{t+1}] + E_t[\tilde{c}_{i,t+1}] - p_t}{p_t}$$

If actual equilibrium conditions deviate from those which were expected, then the return will deviate from its expected value. In the very long run this would not be a problem since expected equilibrium values would probably average out to actual values. However over anything less than the longest run it is certainly possible and even probable that there will be secular changes in equilibrium conditions due to unforseen developments[4] which do not average out. A concrete example of this effect can be found in long term bond yields over the period covered by this study, 1947 to 1965. The quarterly data indicates that over this period the mean of long term yields in each quarter was 3.23% but the average return if long term bonds were sold at the end of each quarter and replaced was -.81%

If term structure effects which cause these holding period yields to differ from yields to maturity can be ignored, this could be decomposed in the following manner:

$$3.23 = \text{Average}\left[\frac{c_{i,t+1}}{p_{it}} + \frac{E_t[\tilde{p}_{i\ t+1} - p_{it}]}{p_{it}}\right]$$

-4.04 = Average [Unexpected Price Change];

$$= \text{Average}\left[\frac{p_{i,t+1} - E_t[\tilde{p}_{i,t+1}]}{p_{it}}\right]$$

-.81% = Net average holding period return.

Even if the term structure effects could not be ignored, it would be foolish to assert that -.81% per annum is an unbiased estimate of the average expected first quarter yield for long term bonds over this period. This analogy is of course directly applicable to stock market returns. Unfortunately yield to maturity data is not available for common stocks, and therefore, the decomposition of returns into expected and unexpected portions may be impossible.[5]

In order to compensate for this source of bias, returns can be viewed in terms of the determinants of equilibrium prices, and then any known deviations of these from their anticipated values could be used to compensate for the unexpected portion of realized returns. This is done by substituting into, in equation (2-1), an expression for the equilibrium price in terms of the parameters of the equilibrium and the attributes of the company. This substitution yields an expression for the random return which is in terms of the random parameters of the future equilibrium, the random future company attributes, and the random cash throw off over the period. A general expression for equilibrium price would be of the form

(2-3) $p_t = p(a_{it}, b_{it}, c_{it}, d_t, e_t, f_t)$

or

$$\tilde{p}_{t+1} = p(\tilde{a}_{i,t+1}, \tilde{b}_{i,t+1}, \tilde{c}_{i,t+1}, \tilde{d}_{t+1}, \tilde{e}_{t+1}, \tilde{f}_{t+1})$$

where a_{it}, b_{it}. and c_{it} are company specific attributes and d_t, e_t and f_t are parameters of the equilibrium at time t. In this case the joint distribution of $\tilde{a}_{i,t+1}$, $\tilde{b}_{i,t+1}$, $\tilde{c}_{i,t+1}$, $\tilde{d}_{i,t+1}$, $\tilde{e}_{i,t+1}$, $\tilde{f}_{i,t+1}$ and $\tilde{c}_{i,t+1}$ will determine the distribution of $\tilde{R}_{it}$. Suppose for example that price can be written

(2-4) $p_{it} = \bar{c}_{it}/\rho_{it}$

where $\bar{c}_{it}$ is the mean cash throw off per unit of time which is expected as of time t, and $1/\rho_{it}$ is the price per unit of mean cash flow which holds for company i at time t. Now assume that asset i represents some physical production process whose productivity follows a martingale. In other words, it is subject to random variations over time, but is always expected to remain at its current level, whatever that may be. Then ρ_{it} is simply the rate at which mean earnings in each future period are capitalized. Furthermore both $\bar{c}_{it}$ and ρ_{it} may change over time so that

(2-5) $\tilde{p}_{i,t+1} = \tilde{\bar{c}}_{i,t+1}/\tilde{\rho}_{i,t+1}$

In (2-5) the probability distributions of all variables are based on information available at time t.

Using these assumptions it is now possible to write a simplified expression for realized returns in terms of the component sources of variation. First define:

$$\delta\tilde{\bar{c}}_{i,t+1} = (\tilde{\bar{c}}_{i\ t+1} - \bar{c}_{it})/\bar{c}_{it}$$

$$\delta\tilde{\rho}_{i,t+1} = (\rho_{i,t+1} - \rho_{it})/\ \rho_{it}$$

$$\delta\tilde{c}_{i,t+1} = (\tilde{c}_{i,t+1} - \tilde{c}_{it})/\tilde{c}_{it}$$

and observe that

$$\tilde{\rho}_{i,t+1} = (1 + \delta\tilde{\rho}_{i,t+1})\rho_{it} \qquad (2\text{-}6)$$

$$\tilde{\bar{c}}_{i,t+1} = (1 + \delta\tilde{\bar{c}}_{i,t+1})\bar{c}_{it} \qquad (2\text{-}7)$$

$$\tilde{c}_{i,t+1} = (1 + \delta\tilde{c}_{i,t+1})\bar{c}_{it}$$

Substituting (2-6), (2-7) and (2-4) into (2-5) yields

$$\tilde{p}_{i,t+1} = p_{it}(1 + \delta\tilde{\bar{c}}_{i,t+1})/(1 + \delta\tilde{\rho}_{i,t+1}) \qquad (2\text{-}8)$$

and substituting this and (2-4) into (2-1) yields

$$\tilde{r}_{it} = (1 + \delta\tilde{\bar{c}}_{i,t+1})/(1 + \delta\tilde{\rho}_{i,t+1}) - 1 + (1 + \delta\tilde{c}_{i,t+1})\rho_{it} \qquad (2\text{-}9)$$

By using the approximation that for $x << 1$

$$\frac{1}{1 + x} \approx 1 - x$$

(2-9) can be written

$$(2\text{-}10) \qquad \tilde{r}_{it} \approx (1 + \delta\tilde{\bar{c}}_{i,t+1})(1 - \delta\tilde{\rho}_{i,t+1}) + (1 + \delta\tilde{c}_{i,t+1})\rho_{it} - 1$$

$$= \delta\tilde{\bar{c}}_{i,t+1} - \delta\tilde{\rho}_{i,t+1} + \rho_{it}(1 + \delta\tilde{c}_{i,t+1}) + \delta\tilde{\bar{c}}_{i,t+1} \cdot \delta\tilde{\rho}_{i,t+1}$$

Equation (2-10) simply says that the realized return is made up of components which come from the percentage change in cash flow expectations ($\delta\tilde{\bar{c}}_{i,t+1}$) the percentage change in the capitalization rate ($\delta\tilde{\rho}_{i,t+1}$), average expected yield portion (ρ_{it}), the return due to deviation of current cash throw off from expected ($\delta\tilde{c}_{i,t+1}\rho_{it}$), and the interaction of rate change with expectations change ($\delta\tilde{\bar{c}}_{i,t+1}\ \delta\tilde{\rho}_{i,t+1}$). It has already been assumed that expectations are unbiased, or that

$$(2\text{-}11) \qquad E_t[\delta\tilde{\bar{c}}_{i,t+1}] = 0.$$

Further simplification will result from the assumption that rates are not expected to change

$$(2\text{-}12) \qquad E_t[\delta\tilde{\rho}_{i,t+1}] = 0$$

and that rate changes and expectation changes are independent

$$(2\text{-}13) \qquad E_t[\delta\tilde{\rho}_{i,t+1}\ \delta\tilde{\bar{c}}_{i,t+1}] = 0$$

Even if independence cannot be assumed (2-13) will hold approximately since the cross product term is a product of two percentage changes which are both typically much less than one. Finally, little error should be imposed by assuming that deviation of the actual cash throw off from its expected value is equal to the change in the mean of the cash generation process. Since the amount by which the actual results fail to satisfy this assumption is multiplied by ρ_{it} this too should be of little concern:

(2-14) $$\delta \tilde{c}_{i,t+1} \approx \delta \tilde{\bar{c}}_{i,t+1}$$

Now equation (2-10) can be rewritten as

(2-15) $$\tilde{r}_{it} \approx (1 + \rho_{it})\, \delta \tilde{\bar{c}}_{i,t+1} - \delta \tilde{\rho}_{i,t+1} + \rho_{it}$$

and taking the expected value of this yields

(2-16) $$\bar{r}_{it} = \rho_{it}$$

This gives the expected return of the financial asset in terms of the determinant of equilibrium price, ρ_{it}. Equation (2-16) is of particular interest because it implies that whatever the equilibrium relationships that determine expected returns may be, they will also apply to capitalization rates under this model of prices.[6]

From equation (2-15) it is possible to compute what the sample average return will be over a number of observations of returns.

$$(2\text{-}16) \qquad \frac{1}{N}\sum_{t=1}^{N} \tilde{r}_{it} = \frac{1}{N}\sum_{t=1}^{N} (1 + \rho_{it})\delta\tilde{\bar{c}}_{i,t+1} - \frac{1}{N}\sum_{t=1}^{N} \delta\tilde{\hat{\rho}}_{i,t+1} + \frac{1}{N}\sum_{t=1}^{N} \rho_{it}$$

As shown above in (2-16) the last term on the right hand side should conform to the market equilibrium relationship. Therefore, when the left hand side is used as a proxy for expected return, there is an implicit assumption that the net contribution of changes in capitalization rates or expectations has been zero. This of course need not be so, even over fairly long averaging intervals as was shown in the case of bonds above.

III Source of β_i

Now that the return has been written in terms of its expectation and the deviations of the components from their anticipated values, it is relatively simple to compute the value of β_i in terms of the joint distribution of these components. The relationships for the market which are analogous to (2-15) and (2-16) are

$$(3\text{-}0) \qquad \begin{aligned} \tilde{r}_{mt} &= (1 + \rho_{mt})\, \delta \tilde{\bar{c}}_{m,t+1} - \delta\tilde{\rho}_{m,t+1} + p_{mt} \\ \bar{r}_{mt} &= \rho_{mt} \end{aligned}$$

Using the definition of covariance and making the further assumption that $0 = E_t[\delta \tilde{\bar{c}}_{m,t+1}\ \delta \rho_{i,t+1}] = E_t[\delta \tilde{\bar{c}}_{i,t+1}\ \delta \rho_{m,t+1}]$

$$\text{Cov}(r_i r_m) = (1 + \rho_{it})(1 + \rho_{mt})\text{Cov}(\delta \tilde{\bar{c}}_{i,t+1} \delta \tilde{\bar{c}}_{m,t+1}) + \text{Cov}(\delta \rho_{i,t+1} \delta \tilde{\rho}_{m,t+1})$$

$$\text{Var}(r_m) = (1 + \rho_m)^2\ \text{Var}\ (\delta \tilde{\bar{c}}_{m,t+1}) + \text{Var}\ (\delta \tilde{\rho}_{m,t+1})$$

so that β_i can be written

$$(3\text{-}1) \qquad \beta_i = \alpha\beta_{cit} + (1 - \alpha)\ \beta_{\rho it}$$

where

$$(3\text{-}2) \qquad \beta_{cit} = \left(\frac{1 + \rho_{it}}{1 + \rho_{mt}}\right) \frac{\text{Cov}\ (\delta \tilde{\bar{c}}_{i,t+1}\ \delta \tilde{\bar{c}}_{m,t+1})}{\text{Var}(\delta \tilde{\bar{c}}_{m,t+1})}$$

$$(3\text{-}3) \qquad \beta\rho_i = \frac{\mathrm{Cov}(\delta\tilde{\bar{\rho}}_{i,t+1}\ \delta\tilde{\bar{\rho}}_{m,t+1})}{\mathrm{Var}(\delta\tilde{\bar{\rho}}_{m,t+1})}$$

$$\alpha = (1+\rho_m)^2\mathrm{Var}(\delta\tilde{\bar{c}}_{m,t+1})/[(1+\rho_m)^2\mathrm{Var}(\delta\tilde{\bar{c}}_{m,t+1}) + \mathrm{Var}(\delta\tilde{\rho}_{m,t+1})]$$

Equation (3-1) shows that the anticipated β_i is a weighted average of a cash flow expectations coefficient and a capitalization rate coefficient, with the weights being determined by the fraction of total variance of market return attributable to each source. These could easily change over time to reflect prevailing attitudes and uncertainties. An historically based estimate of β_i could be perfectly right on average and still fail to be a good proxy for the anticipated values, because changing sources of uncertainty cause the anticipated values of β_i to fluctuate around their average historical values. The historical estimate could still be an appropriate predictor of average future values.

Since β_i, which is the overall regression coefficient of r_i, return to asset i, on r_m, return to the market portfolio, is a weighted average of the two regression coefficients of r_i on the component sources of r_m, if one of these is larger than β_i then the other must be smaller. If for a particular asset i, the sensitivity of its capitalization rate to a change in the market rate is greater than β_i, then when the market return is due to a shift in the capitalization rate it should show a larger change in value than would be predicted by using β_i in a single factor model. When the capitalization rate sensitivity is greater than β_i, the cash flow sensitivity must be less than β_i. Therefore, when market return is due a shift in cash flow expectations security i will have a change in value of smaller magnitude than would be predicted by using β_i.

The deviation of the actual return r_i from that predicted by the single factor β_i model is measured as the residual error. In the paragraph above it has been shown that these residual errors can be expected to depend on the source of the market return. If it can be demonstrated that the sensitivity coefficients of individual returns to the sources of market return are themselves functions of β_i, then two important conclusions will result: 1) The variance of the residual error term for a portfolio need not tend towards zero with greater diversification, and 2) the residual error term will be a function of β_i and of the relative magnitude of the sources of market return. When this second statement is true, cross sectional studies (which by their nature hold the relative magnitude of the sources of market return constant) should show a dependence of residual error on β_i when looking at realized rather than expected returns. To pursue this development, the definition of the capitalization rate sensitivity along with the fact that capitalization rates conform to the market equilibrium relationship can be used to relate the rate coefficient for a given security to its overall β coefficient

Define:

$$\beta_i = \mathrm{Cov}(\tilde{r}_{it}, \tilde{r}_{mt})/\mathrm{Var}(\tilde{r}_{mt})$$

(3-3) $\quad \beta\rho_i = \mathrm{Cov}(\delta\tilde{\rho}_{it}, \delta\tilde{\rho}_{mt})/\mathrm{Var}(\delta\tilde{\rho}_{mt})$

(3-2) $\quad \beta c_i = \mathrm{Cov}(\delta\tilde{\bar{c}}_{it}, \delta\tilde{\bar{c}}_{mt})/\mathrm{Var}(\delta\tilde{\bar{c}}_{mt})$

$\tilde{r}_i$ = actual return to security i;

ρi = expected return to security i = capitalization rate;

ρm = expected return on market portfolio or capitalization rate for national wealth;

ρ_n = risk free rate;

$\delta\tilde{\rho}_{nt}$ = fractional change in ρ_n for period starting at t;

(3-4) $\beta^m_n = \mathrm{Cov}(\delta\tilde{\rho}_{nt}, \delta\tilde{\rho}_{mt})/\mathrm{Var}(\delta\tilde{\rho}_{mt})$

Now recall that substitution of (2-16) into (1-1) implies that

(3-5) $\quad \rho_i = \rho_n(1 - \beta_i) + \beta_i(\rho_m)$

In order to find $\beta\rho_i$ as a function of β_i, (3 - 5) can be used to determine $\Delta\rho_i$ in terms of $\Delta\rho_m$, β_i, and $\Delta\rho_n$, then $\beta\rho_i$ will result from substitution in (3-3). From (3-6)

$$\Delta\rho_{it} = \Delta\rho_{nt}(1 - \beta_i) + \beta_i\Delta\rho_{mt}$$

and

(3-6) $\delta\tilde{\rho}_{it} = \Delta\rho_{it}/\rho_{it} = (1 - \beta_i)(\Delta\rho_n/\rho_i) + \beta_i(\Delta\rho_m/\rho_i)$

Now multiplying by $\Delta\rho_m/\rho_m$ yields

$$(\Delta\rho_n/\rho_i)(\Delta\rho_m/\rho_m) = (1 - \beta_i)(\Delta\rho_m/\rho_m)(\Delta\rho_n/\rho_n)(\rho_n/\rho_i) + \beta_i(\Delta\rho_m/\rho_m)^2\rho_m/\rho_i.$$

Taking the expectation and then dividing by $Var(\Delta\rho_m/\rho_m)$ will give $\beta\rho_i$ by (3-3). Making use of the definition of β_n^m in (3-4) above:

$$\beta\rho_i = (1 - \beta_i)\beta_n^m \rho_n/\rho_i + \beta_i \rho_m/\rho_i$$

In order to simplify this expression rewrite it as

$$\beta\rho_i = (1 - \beta_i)(1 + \beta_n^m) \rho_n/\rho_i + \beta_i \rho_m/\rho_i - (1 - \beta_i) \rho_n/\rho_i$$

$$= (1 - \beta_i) \rho_n/\rho_i + \beta_i \rho_m/\rho_i + \beta_n^m(1 - \beta_i) \rho_n/\rho_i - (1-\beta_i)\rho_n/\rho_i$$

Notice that by the equilibrium condition (3-5) the first two terms on the r.h.s. are equal to ρ_i/ρ_i or one. Therefore

$$\beta\rho_i = 1 - (1 - \beta_n^m) (1 - \beta_i) \rho_n/\rho_i$$

or

$$(1 - \beta\rho_i) = (1 - \beta_i)(1 - \beta_n^m) \rho_n/\rho_i \qquad (3-7)$$

Two attributes of this relationship are of importance:

1) when β_i is exactly equal to one then $\beta\rho_i$ will also be equal to one, and 2) when β_i is different from one, $\beta\rho_i$ will probably

differ from one by a smaller amount. The second statement will always be true when $(1 - \beta_n^m)\rho_n/\rho_i$ is less than one. Recalling that β_n^m can be considered as a regression coefficient of fractional changes in the risk free rate on fractional changes in the capitalization rate for risky wealth, there seems to be no reason to expect (ex ante) that this will take on a negative value.[7] This being the case the term $(1 - \beta_n^m)$ will be less than or equal to one. For all β_i greater than zero, the term ρ_n/ρ_i will be less than one so that in general $(1 - \beta_n^m)\ \rho_n/\rho_i$ can be expected to be less than one, and the relationship between βo_i and β_i will be as depicted in Figure 1. This analysis can be viewed as applying either to the ex ante relationship between β_i, β_{ci}, $\beta\rho_i$ and β_n^m or their observed values, therefore, any deviation of the path of the risk free rate from its expected course will influence the observed β_i values through the sample value of β_n^m.

Since the capitalization rate sensitivity is a function of β_i, it should be true that in any period of time when market returns are due predominantly to a single source, the errors of the single factor model in explaining asset returns should also be a function of β_i. To follow this argument diagrammatically consider the relationship between capitalization rate sensitivity and β_i shown in Figure 1.

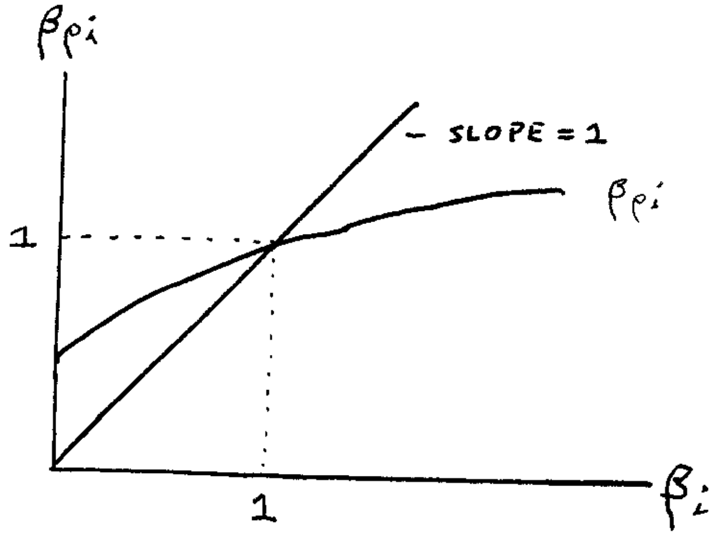

FIGURE 1

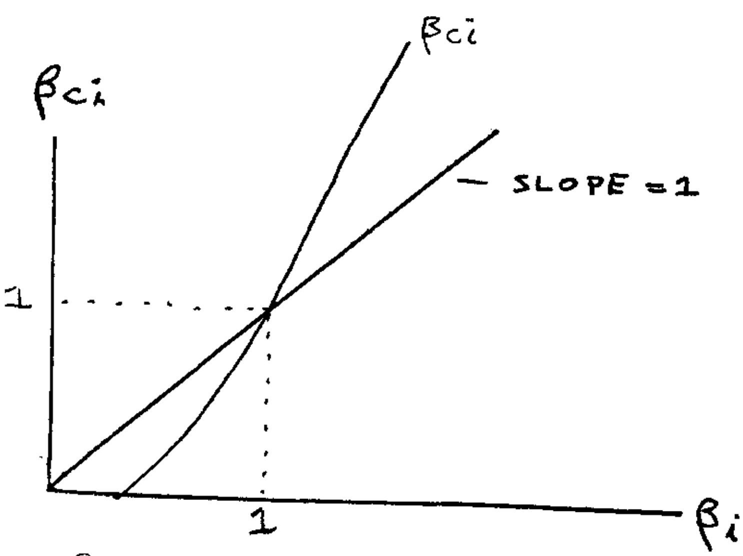

FIGURE 2

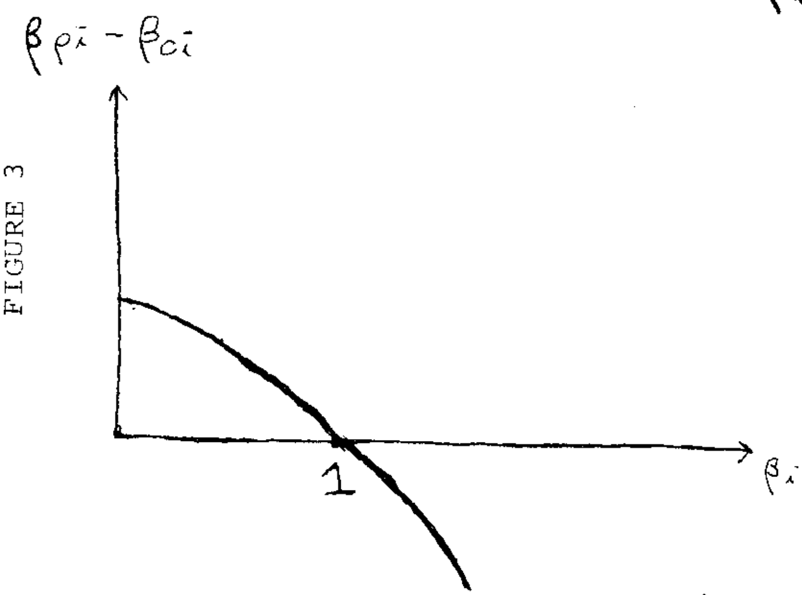

FIGURE 3

When β_i is less than 1, βo_i is larger than β_i, and when β_i is greater than 1, βo_i is less than β_i. This corresponds to β_n^m positive in equation (3-7). Since β_i is a weighted average of the two different sensitivities, βo_i and βc_i, the relationship between βc_i and β_i is determined by equation (3-1) or equivalently (3-8)

(3-1) $\beta_i = \alpha\beta o_i + (1 - \alpha)\, \beta_{ci}$

(3-8) $\beta_{ci} = \beta_i + \frac{\alpha}{1-\alpha}\, (\beta_i - \beta o_i)$

It is interesting to note at this point that equation (3-1) determines each β_i in terms of the market wide variables $Var(\Delta o_m/o_m)$ and $Var(\Delta c_m/c_m)$ and the individual company attributes $\beta\rho_i$ and βc_i. However, substitution for βo_i from equation (3-7) which relates βo_i to β_i and β_n^m in principal yields an expression for β_i as a function of the single company specific variable βc_i and the market wide variables β_n^m, $Var(\Delta\rho_m/\rho_m)$, and $Var(\Delta c_m/c_m)$.[8] In other words when βc_i is known for the company, market wide conditions determine βo_i and β_i. This might be of particular value in calculating a required rate of return for purposes of capital budgeting or regulation from the joint probability distribution of cash flows to the firm and the economy.

IV. Relationship to the Single Factor Model

By starting with the two factor model of equation (2-15) and rotating the factors, an expression showing the systematic nature of residual errors and the single factor model can be derived.

$$\text{(2-15)} \qquad \tilde{r}_{it} = o_{it} + (1 + \rho_{it})\delta\tilde{\bar{c}}_{i,t+1} - \delta\tilde{\rho}_{i,t+1} + \tilde{e}_{it}$$

By using equations (3-0), (3-2) and (3-3) $\tilde{r}_i$ can be separated into systematic components and an unsystematic error term.

$$\text{(4-1)} \qquad \tilde{r}_{it} = \rho_n + \beta_i(\rho_{mt} - \rho_{nt}) + (1 + \rho_m)\ \beta_{ci} - \delta\tilde{\bar{c}}_{m,t+1} + \beta\rho_i\ \delta\tilde{\rho}_{m,t+1} + \tilde{e}_{it}$$

This can be broken down into a constant, a term equal to $\beta_i\tilde{r}_{mt}$, and a residual which, although independent of r_m, depends on β.

Consider the following analogous situation

$$\text{(4-2)} \qquad R_m = R_1 + R_2$$

$$\text{(4-3)} \qquad R_i = \beta_1 R_1 + \beta_2 R_2$$

Suppose for some reason we are interested in estimating

$$\text{(4-4)} \qquad R_i = \beta\ R_m + e$$

where

$$\text{(4-5)} \qquad \beta = W\beta_1 + (1 - W)\beta_2$$

Substituting (4-2) and (4-5) into (4-4) yields

(4-6) $R_i = [W\beta_1 + (1 - W)\beta_2](R_1 + R_2) + e$

but (4-3) can be broken down as shown below into the same form as (4-6) to get an explicit expression for e.

$$(4\text{-}3') \qquad R_1 = W\beta_1(R_1 + R_2) + (1 - W)\beta_1 R_1 - W\beta_1 R_2 + (1 - W)\beta_2(R_1 + R_2) - (1 - W)\beta_2 R_1 + W\beta_2 R_2$$

and this can be simplified by combining terms to

$$(4\text{-}7) \qquad R_1 = (W\beta_1 + (1 - W)\beta_2)(R_1 + R_2) + (\beta_1 - \beta_2)((1 - W)R_1 - WR_2)$$

by comparing (4-7) to (4-6)

$$e = (\beta_1 - \beta_2)((1 - W)R_1 - WR_2)$$

In equation (4-1) the same type of separation can be performed:

From (3- 1)

$$\beta_i = \alpha\beta_{ci} + (1 - \alpha)\beta\rho_i$$

This can be used with equation (4-1) to find equation (4-9) which is analogous to (4-3') above.

$$(4\text{-}1') \qquad \tilde{r}_{it} = \rho_n + \beta_i(\rho_{mt} - \rho_{nt}) + (\alpha\beta_{ci} + (1-\alpha)\beta_{\rho i})(\delta\tilde{\bar{c}}_{mt}(1+o_m) - \delta\tilde{o}_{mt}) + (\beta o_i - \beta c_i)(-\alpha\delta\tilde{o}_{mt} - (1 - \alpha)\delta\tilde{\bar{c}}_{mt}[1 + \rho_{mt}]) + \tilde{e}_{it}$$

(4-8) $$\tilde{r}_{it} = \rho_n(1-\beta_i) + \beta_i(\rho_m + (1+\rho_m)\delta\tilde{\bar{c}}_t - \delta\tilde{\rho}_{mt}) + (\beta\rho_i - \beta c_i)$$
$$(-\alpha\ \delta\tilde{\rho}_{mt} - (1-\alpha)\delta\tilde{\bar{c}}_{mt}(1+\rho_{mt})) + \tilde{e}_{it}$$

(4-9) $$= \rho_n(1 - \beta) + \beta_i\tilde{r}_{mt} + (\beta\rho_i - \beta c_i)(-\alpha\ \delta\tilde{\rho}_{mt} - (1-\alpha)$$
$$(1 + \rho_{mt})\delta\tilde{\bar{c}}_{mt}) + \tilde{e}_{it}$$

The residual error term in the observed relationship between security returns and market return contains components which depend on the nature of the economic climate over the estimation period. While the ex ante expectation of these terms is zero since a martingale type process has been assumed, the observed data will probably not have a zero mean. There are three cases where this can be ignored. They are when one of the three following conditions is met

a) Both β components equal $\beta\rho_i = \beta c_i = \beta_i$

b) No variation in rates $\beta\rho_i = 0$, $\beta c_i = \beta_i$

c) No variation in expectations $\beta c_i = 0$, $\beta\rho_i = \beta_i$

The last two cases are simply single factor models. The first represents the case where it is impossible to identify the factors from the data and hence a single factor model is also sufficient.

Equation (4-9) along with the relationship between βc and $\beta\rho$ shown in figures 1 and 2 indicates how residuals should be related cross sectionally in any period. Figure 3 shows the value of $\beta\rho_i - \beta c_i$

plotted versus β_i. The term $(-\alpha\, \delta\tilde{\sigma}_{mt} - \delta\bar{c}_{mt}(1 + \rho_m)(1 - \alpha))$ will determine the sign and magnitude of the factor which multiplies the curve in figure 3 to produce residual errors. (See Appendix I.) This multiplier will be positive if market return is above normal and disproportionately due to capitalization rate changes or if market return is below average due to an exceptional decrease in expectations. The sign of the multiplier will be negative if negative market returns are caused predominantly by rate changes or positive market returns are caused by changes in expectations. This analysis will apply equally well to an average of successive values of equation (4-9). This means that the observed constant terms or mean excess returns estimated over an interval will have a component which is the average of these errors and will show a non-linear relationship to β determined by the curve of figure 3 and the fraction of average market return which comes from each source.

Conversely by observing the slope of the residual error terms of the single factor model and the deviation around its mean it should be possible to identify the dominant source of market return. Positive slope and market return above its mean should be associated with an upward revision of expectations. Negative slope and market return below the mean should be associated with a downward revision of expectations. This is so because the expectations beta, βc_i, is greater than β_i for high β_i securities, and less than β_i for low β_i securities (this assumes that β_n^m is non-negative).

Similarly, negative slope from the regression of residual on β_i in combination with market return above the mean should be associated with positive returns from a downward revision of the required rate of return to risky investments. A positive slope of residual vs. β_i and market return below average should be associated with an upward revision of the capitalization rate for risky wealth. This results from the fact that $\beta\rho_i$ is less than β_i for high β_i securities and greater than β_i for low β_i securities.

The implied relationship between residual errors, β, and the source of market returns results in two testable hypotheses. The first is that cross sectionally there should be a strong association between the residual errors of a single factor model and β whenever returns from the two sources do not balance out. The second is that the nature of this relationship should be associated with the dominant source of overall market returns. This hypothesis is predicated on the validity of the capital asset pricing model as applied to capitalization rates, and if confirmed would have to be interpreted as implying an association between β_i and required rates of return.

V. Empirical Validation

The preceeding analysis has generally not specified whether the random variables, variances, and covariances describe attributes of the data prior to the event or represent realized sample values. It has in fact been unnecessary to make this distinction until now, since the relationships hold for both.

The one exception to this is the ρ term in the capital asset pricing model which is an <u>ex ante</u> value. Observed β terms will differ from their <u>ex ante</u> values if β_n^m, $\mathrm{Cov}(\delta\tilde{\rho}_{it},\delta\tilde{\rho}_{mt})$, $\mathrm{Cov}(\delta\tilde{\bar{c}}_{it}, \delta\tilde{\bar{c}}_{mt})$, $\mathrm{Var}(\delta\tilde{\bar{c}}_{mt})$, or $\mathrm{Var}(\delta\tilde{\sigma}_{mt})$ differ from their anticipated values. In all these possible cases the difference between anticipated and observed values of β_i will be a single function of β_i for the entire sample period. This can be seen by drawing curves similar to figures 1, 2, and 3 for the <u>ex ante</u> and observed values of the variables in question and noting the difference in the values of β_i implied for a firm with any particular value of βc_i. Since this paper is concerned with the inappropriateness of <u>ex ante</u> estimates to observed data, provision should be made to eliminate as many sources of deviation as possible. By using sample period values for the β_i's, the sample values of βc_i, β_n^m, and $\beta\rho$ for the period must apply on average to both the β_i values and the return data, eliminating possible bias from these sources. Any deficiencies of realized data in testing the <u>ex ante</u> capital asset pricing model can only be increased when the sample means of these values are also allowed to deviate from their anticipated value.

The sum of all errors of the market model for a given security over the estimation period must be zero. Nevertheless, the multiple source model implies that in any subperiod in which market return was due disproportionately to one of the sources, the single factor market model will produce errors which are correlated with the full period β_i. In subperiods when market returns were generated by both sources in approximately the same proportion as for the full period of β estimation, the errors should show no relationship with the β_i. This suggests a test for the possibility that the single factor model is inadequate to explain relative performance of securities over time. If no period can be found in which there is a strong association between errors of the single factor model and the β coefficients, then it is unlikely that the model using differentiated sources of market return will lead to an improved explanation of returns. Unfortunately simply finding a significant relationship between the errors and β_i's in subperiods is only a necessary condition for this theory to be valid, but it is not sufficient for confirmation since such a relationship could arise from a misspecification of $\tilde{r}_m$ or r_f as easily as from the mechanisms derived above.

To perform the test, β coefficients must be estimated over a period of time for a number of assets. Then the residual errors for this set of assets can be examined at specific times within the overall estimation period. Because of the random nature of observed β's only ex post values were used. The general procedure used here was to estimate a coefficient for each asset, then to form portfolios of assets covering 25 different ranges of security β coefficients. The residual errors of these portfolios, or deviation of actual return from predicted return using actual market return, serve as the data for the test.

The market equilibrium condition

$$(5\text{-}1) \qquad \bar{r}_{it} = r_{ft} + \beta_i(\bar{r}_{mt} - r_{ft})$$

in combination with a single market factor model of security returns

$$(5\text{-}2) \qquad \tilde{r}_{it} = a_i + b_i\, \tilde{r}_{mt} + \tilde{\epsilon}_{it}$$

imply that the individual β_i should be estimated by regressing excess return on a security (above the riskless rate) on excess return for the market without including a constant term in the regression. To see this take the expectation of equation (5-2).

$$\bar{r}_{it} = a_i + b_i\, \bar{r}_{mt}$$

but from (5-1)

$$\bar{r}_i = (1 - \beta_i)\, r_{ft} + \beta_i \bar{r}_m$$

and from (5-2) and the definition of r_{mt}

$$\beta_i \approx b_i$$

thus

$$a_i = (1 - \beta_i)\ r_f$$

$$= (1 - b_i)\ r_f$$

and (5-2) becomes

$$(5\text{-}3) \qquad (\tilde{r}_{it} - r_{ft}) = b_i(\tilde{r}_{mt} - r_{ft}) + \tilde{\epsilon}_{it}$$

which implies that the appropriate way to estimate the b_i terms is without a constant term.

Consideration of taxation and market imperfections such as transaction costs could give rise to deviation of observed equilibrium conditions from the ideal case of equation (5-1). These deviations might be adequately compensated for by the inclusion of a constant term in the equilibrium condition and hence in the estimation of the β_i.

Tests were performed using both procedures to estimate the β_i. Since residuals were to be examined for annual and quarterly subperiods of the entire estimation period, the constant terms were computed as shown in equation (5-4), and in an analogous manner for the quarterly data. This insures that the sum of residuals over the entire period will be zero.

$$(5\text{-}4) \qquad a_i = \frac{1}{(19)} \sum_{t=0}^{18} \prod_{n=1}^{12} (1 + \tilde{r}_{i,12t+n}) - \beta_i \sum_{t=0}^{18} \prod_{n=1}^{12} (1 + r_{m,12t+n})$$

Annual data was used for the tests so that the individual periods could be associated easily with other economic phenomena. To gain more observations quarterly data for the same period were also used.

The results of regressions of residual errors in annual portfolio returns against portfolio β's are shown in Tables 1 and 2 for the two types of β estimate. Table 1 represents the case where β is estimated with the inclusion of a constant term in the market relationship. Table 2 is based on estimates of β made in strict accordance with the pure equilibrium model, without a constant term. Table 4 shows similar results for the quarterly data over the same period. All quarterly data β estimates were made with a constant term included in the regression

The test with the constant term included in the estimation of β must be considered most stringent since the omission of a constant if needed could cause a relationship similar to the type observed. Table 1 shows that in 11 of the 19 years there was a significant relationship between β and the errors. A simple bias in estimation of the β's would account for such a relationship only if the slope coefficients were related to the actual return on the market. This possible source of the extremely positive test results can be tested for by regressing the slope coefficients on the actual market return for the corresponding year. The results are shown in Table 3. Clearly in all cases the systematic influences on the individual security returns which are not related to the size of market return far outweight those which are.

For the quarterly data 42 out of 76 quarters are significant at the 5% level or better and 20 of these are significant at the 1% level or better. The probability of observing this by chance is less than 10^{-72} assuming serial independence of returns.

This evidence indicates that a single factor model is not adequate to describe portfolio returns. It should be clear that a single factor model of returns is adequate only if it is sufficient to adjust the _ex post_ returns for the realized values of each common factor for the sample. In general it will be necessary to use one factor to adjust the realized data for each of the independent factors generating it, since the sample mean of these factors will be random variables. This does not imply anything about the minimum number of _ex ante_ risk measures, however, since one may still be adequate to differentiate expected returns. In terms of the model of market returns from multiple sources the interpretation of these data is straightforward. There are four cases to consider corresponding to the possible combinations of the signs of market return and the slope of errors vs. β.

From the relationship of individual returns posited earlier, if market return is due to changing cash flow expectations then high β securities will change more than expected whereas low β ones will change less. If the change in expectations produces a positive return, then the high β securities will have positive residuals and the low β securities will have negative residuals, thus the market return above the mean combined with a positive slope of errors vs. β would be the observed result. If the market movement were predominantly caused by changes in cash flow expectations then a positive market return would be associated with a positive slope of residual errors regressed on β.

Alternatively, if the market return is caused predominantly by changes in the capitalization rate, then when market return is positive the high β securities will have a negative residual errors and the low β securities will have positive errors. Conversely, when the market return is negative, the high β securities will not move so far as expected and hence the residual errors will be positive whereas the low β securities will move farther than expected, giving rise to negative residuals. Table 5 shows these relation-

TABLE 1: ESTIMATES OF β MADE WITH A CONSTANT

	YEAR	SLOPE	T-TEST	MEAN RES	R-SQUARED	MARKET	DOMINANT SOURCE[1]
	1947	0.0421	1.417	0.00056	0.0669	-0.0112	rate (+)
	1948	0.0395	1.598	-0.00275	0.0836	-0.0254	rate (+)
*	1949	-0.2998	-9.290	-0.00349	0.7550	0.2236	rate (-)
*	1950	0.2752	11.159	-0.00285	0.8164	0.3610	cash (+)
*	1951	-0.0985	-3.222	-0.00367	0.2705	0.1443	cash (-)
*	1952	-0.0876	-3.770	-0.00159	0.3367	0.0926	cash (-)
*	1953	-0.1092	-5.568	0.00496	0.5255	-0.0399	cash (-)
	1954	0.0723	2.030	0.00098	0.1283	0.5775	cash (+)
	1955	0.0346	1.359	-0.00618	0.0619	0.1891	cash (+)
	1956	0.0559	1.489	0.00519	0.0734	0.0592	rate (+)
*	1957	-0.1178	-5.704	0.01126	0.5375	-0.1634	cash (-)
	1958	-0.0044	-0.108	-0.00095	0.0004	0.5840	rate (-)
	1959	0.0216	0.520	0.00099	0.0096	0.1314	rate (+)
*	1960	-0.1805	-7.631	0.00427	0.6753	-0.0482	cash (-)
*	1961	-0.2018	-7.301	-0.00850	0.6556	0.2797	rate (-)
*	1962	0.1074	5.293	-0.00110	0.5001	-0.1418	rate (+)
*	1963	0.1029	3.645	0.00073	0.3218	0.1850	cash (+)
	1964	0.0250	0.602	-0.00475	0.0128	0.1698	cash (+)
*	1965	0.3232	9.340	0.00690	0.7570	0.2323	cash (+)

[1] See Table 4

*Significant at 1% level or better

TABLE 2: ESTIMATES OF β MADE WITHOUT A CONSTANT

YEAR	SLOPE	T-TEST	MEAN RES	R-SQUARED	MARKET	DOMINANT SOURCE
1947	0.0162	0.623	0.00005	0.0137	-0.0112	rate (+)
1948	0.0435	2.010	-0.00332	0.1261	-0.0254	rate (+)
* 1949	-0.1405	-3.809	-0.00283	0.3413	0.2236	rate (-)
* 1950	0.1830	4.211	-0.00151	0.3877	0.3610	cash (+)
1951	0.0004	0.015	-0.00340	0.0000	0.1443	rate (+)
1952	-0.0015	-0.082	-0.00157	0.0002	0.0926	cash (-)
1953	-0.0045	-0.213	0.00432	0.0016	-0.0399	cash (-)
* 1954	0.0823	2.523	0.00340	0.1852	0.5775	cash (+)
1955	0.0202	0.676	-0.00569	0.0160	0.1891	cash (+)
* 1956	0.0865	3.740	0.00504	0.3331	0.0592	rate (+)
* 1957	-0.0710	-4.204	0.01001	0.3896	-0.1634	cash (-)
* 1958	-0.1970	-4.452	0.00149	0.4145	0.5840	rate (-)
1959	0.0338	0.864	0.00119	0.0259	0.1314	rate (+)
1960	-0.0182	-0.716	0.00359	0.0180	-0.0482	cash (-)
1961	-0.0579	-1.476	-0.00756	0.0722	0.2797	rate (-)
1962	0.0162	0.584	-0.00225	0.0120	-0.1418	rate (+)
1963	-0.0395	-1.200	0.00120	0.0489	0.1850	rate (-)
1964	-0.0617	-2.016	-0.00436	0.1267	0.1698	rate (-)
* 1965	0.1416	3.077	0.00760	0.2527	0.2323	cash (+)

*Significant at 1% level or better

TABLE 3

REGRESSION OF SLOPES ON R_m

$$Slope_t = a + b(R_{mt}) + e_t$$

	For all years	For significant years	For not significant years
R_m coefficient	.21212	.1731	-.08525
Standard error	-.14488	.2559	.22141
Constant	-.02701	.03573	-.03457
Standard error	.03622	.09379	.03136
R^2	.1120	.0833	.0146
F Statistic	(1,17)=2.1435	(1,5)=.4575	(1,10)=.1432
Number of observations	19.	7	12

TABLE 4: 25 CO PORTFOLIOS-REGRESSION OF MEAN QUARTERLY ERRORS AGAINST MEAN BETAS WHERE BETAS COMPUTED WITH CONSTANT IN ORIG. REGRESSIONS

[illegible]	Year	Slope	TStat	R Squared	Market Return	Sign of Return minus Aver. Return	Source	Sign of Source
[illegible]	1947	0.0[illegible]	1.[illegible]	0.10796	-.01[illegible]0	-		
[illegible]	1947	-.02[illegible]	-1.6[illegible]4	0.10[illegible]0	-.0687	-		
[illegible]	1947	0.0[illegible]0	[illegible].0[illegible]	0.[illegible]275	0.0[illegible]	+	Cash	+
[illegible]	1947	0.0179	1.0[illegible]	0.04[illegible]	0.01[illegible]7	+		
[illegible]	1948	0.0551	[illegible].[illegible]13	0.[illegible]	0.00[illegible]	+	Cash	+
[illegible]	1948	-.0247	-[illegible]	0.0[illegible]	0.1098	+		
[illegible]	1948	0.01[illegible]	[illegible].7[illegible]6	0.1[illegible]211	-.100[illegible]	-		
[illegible]	1948	-.00[illegible]	-0.[illegible]5	0.0103[illegible]	[illegible].0[illegible]	-		
1	1949	-.04[illegible]	-[illegible].0969	0.29[illegible]	0.0[illegible]	+	Rate	-
[illegible]	1949	-.079[illegible]	-5.7[illegible]	0.[illegible]	-.[illegible]0	-	Cash	-
[illegible]	1949	-.0[illegible]27	-[illegible].7073	0.[illegible]7[illegible]0[illegible]	[illegible].152[illegible]	+	Rate	-
[illegible]	1949	-.02[illegible]	-[illegible]	0.[illegible]73	[illegible].116[illegible]	+	Rate	-
1	1950	0.012[illegible]	0.[illegible]	0.0[illegible]8	0.0485	+		
[illegible]	1950	0.06[illegible]	[illegible].1221	0.5[illegible]	-.00[illegible]	-	Rate	+
[illegible]	1950	0.0349	2.[illegible]	0.1000[illegible]	0.1571	+	Cash	+
[illegible]	1950	0.107[illegible]	[illegible].[illegible]	0.746[illegible]	0.[illegible]	+	Cash	+
1	1951	-.0596	-6.[illegible]	0.61564	[illegible].0[illegible]71	+	Rate	-
[illegible]	1951	-.0095	-0.7[illegible]	0.0[illegible]67	-.0[illegible]	[illegible]		
[illegible]	1951	0.0[illegible]	[illegible].[illegible]	0.[illegible]	0.[illegible]	+		
[illegible]	[illegible]	-.0[illegible]1	-1.[illegible]	0.07706	-.00[illegible]	-		
[illegible]	[illegible]	-.0[illegible]	-[illegible]	0.[illegible]	0.0[illegible]	+	Rate	-
[illegible]	[illegible]	-.0[illegible]1	-[illegible]	0.[illegible]	0.0[illegible]	+		
[illegible]	[illegible]	-.0[illegible]	-[illegible]	0.[illegible]	-.0[illegible]	-	Cash	[illegible]
[illegible]	[illegible]	-.01[illegible]	-1.[illegible]	0.0[illegible]	0.0[illegible]	+		
[illegible]	[illegible]	-.0[illegible]	-[illegible]	0.1[illegible]70	0.0[illegible]	+		
[illegible]	[illegible]	[illegible]	[illegible]	0.[illegible]	-.0[illegible]	[illegible]	Rate	+

Quarter	Year	Slope	T-Stat	R Squared	Market Return	Sign of Return minus Av. M. Return	Source	Sign of Source
3	1953	-.0506	-6.4756	0.64295	-.0717	-	Cash	-
4	1953	-.0663	-3.9028	0.[illegible]	0.0774	+	Rate	-
1	1954	-.0[illegible]	-1.2279	0.06151	0.1127	+		
2	1954	-.0204	-1.6281	0.10271	0.0689	+		
3	1954	-.0[illegible]	-7.[illegible]	0.74605	0.1009	+	Rate	-
4	1954	0.0365	6.[illegible]	0.66971	0.1720	+	Cash	+
1	1955	0.0359	2.2366	0.19654	0.0522	+	Cash	+
2	1955	-.0190	-1.4844	0.09741	0.0730	+		
3	1955	-.0049	-0.4249	0.00779	.0014	+		
4	1955	0.0002	0.0157	0.00003	0.0509	+		
1	1956	0.0058	0.3964	0.00755	0.0356	+		
2	1956	0.0156	1.3274	0.07209	-.0249	-		
3	1956	0.0263	1.0571	0.14251	-.0227	-		
4	1956	0.0131	0.6[illegible]	0.01999	0.0275	+		
1	1957	-.0519	-2.5806	0.22417	-.0091	-	Cash	-
2	1957	0.0134	0.7762	0.02572	0.0707	+		
3	1957	-.0572	-2.9476	0.28066	-.1079	-	Cash	-
4	1957	-.0898	-7.5847	0.71169	-.1070	-	Cash	-
1	1958	-.0509	-2.9100	0.26931	0.1219	+	Rate	-
2	1958	-.0266	-2.1926	0.17178	0.1076	+	Rate	-
3	1958	0.0570	7.7601	0.59515	0.1737	+	Cash	+
4	1958	-.0110	-1.3436	0.06748	0.1055	+		
1	1959	0.0153	1.0442	0.04526	0.0727	+		
2	1959	0.0490	2.2606	0.12130	0.0269	+	Cash	+
3	1959	-.0110	-1.1041	0.05037	-.0458	-		
4	1959	0.0361	1.5906	0.11054	0.0457	+		
1	1960	-.0026	-1.2094	0.00189	-.0663	-		

Quarter	Year	Slope	TStat	R Squared	Market Return	Sign of Return minus Aver. Return	Source	Sign of Source
2	1960	-.0329	-6.2721	0.63105	0.0156	+	Rate	-
3	1960	-.0346	-2.5860	0.22526	-.0515	-	Cash	-
4	1960	-.0905	-5.7149	0.58677	0.0519	+	Rate	-
1	1961	-.0017	-0.0718	0.00022	0.1937	+		
2	1961	-.0064	-0.3921	0.00664	-.0011	+		
3	1961	-.0731	-5.1281	0.53344	-.0069	-	Cash	-
4	1961	-.0596	-4.2771	0.44701	0.0556	+	Rate	-
1	1962	0.0333	2.7704	0.25021	-.0085	-	Rate	+
2	1962	0.1217	11.8000	0.85823	-.2390	-	Rate	+
3	1962	-.0255	-2.6771	0.23757	0.0166	+	Rate	-
4	1962	-.0361	-2.5097	0.21498	0.0907	+	Rate	-
1	1963	0.0060	0.5490	0.01293	0.0732	+		
2	1963	0.0273	2.4233	0.20339	0.0463	+	Cash	+
3	1963	0.0093	0.7277	0.02251	0.0090	+		
4	1963	0.0661	6.0486	0.61400	0.0057	+	Cash	+
1	1964	0.0785	4.1944	0.43740	0.0664	+	Cash	+
2	1964	-.0182	-1.0680	0.04725	0.0144	+		
3	1964	0.0031	0.2281	0.00226	0.0448	+		
4	1964	-.0356	-2.3825	0.19794	.0013	+	Rate	-
1	1965	0.0362	3.0488	0.28782	0.0800	+	Cash	+
2	1965	0.0428	4.1718	0.43075	-.0595	-	Rate	+
3	1965	0.0696	5.3191	0.55160	0.0955	+	Cash	+
4	1965	0.1414	11.2804	0.84692	0.0965	+	Cash	+

TABLE 5: Association of Signs of Slope and Market Return with Sources of Return. When any two signs are known, the Third is Implied.

Case	Slope of res v.s. β	Market Return	Source	Years from Table 1	Years from Table 2
I	+	+	Cash +	1950* 1954, 1955 1963*, 1964 1965*	1950*, 1954* 1955, 1965,
II	-	-	Cash Flow -	1951, 1952 1953*, 1957* 1958, 1960*	1952, 1953 1957*, 1960
III	-	+	Rate Change -	1949* 1958 1961*	1949*, 1958*, 1961, 1963 1964
IV	+	-	Rate Change +	1947 1948 1962* 1956 1959	1947, 1948, 1951, 1962, 1956*, 1959

ships qualitatively and the years in which they occur for the two sources of return.

Having confirmed the posited relationship between residual errors of the single factor market model and β_i, it is now possible to test for the agreement between the source and sign of market returns and the slope of the residuals versus β_i. Since neither ρ_{mt} nor $\bar{c}_{mt}$ is observable, other variables must be substituted which are both observable and related to the quantities of interest.

The capitalization rate ρ_{mt} in this model can be viewed as an average over the long run of required future rates of return on risky assets. It is analogous to the long term bond yield. When changes in the long term bond yield are induced by changes in time preferences then changes in ρ_{mt} will be in the same direction. However, when changes in the long term yield or ρ_{mt} are caused by shifting risk preferences or changes in the perceived riskiness of the market portfolio, then the two variables will move in opposite directions. Still it is reasonable to expect the former type of relationship to provide a strong enough association so that the long term yields can be used as a proxy for the variable ρ_m.

For $\bar{c}_{mt}$, corporate profit expectations, the assumption has been made that when corporate profits increased by more than the average for the entire period, that expectations were revised upwards.

When corporate profits fell, or rose by less than the mean, it has been assumed that expectations were revised downwards. The association between this variable derived from corporate profits and the changes in expectations is subject to considerable error. The *ex post* sample mean is clearly an arbitrary number when viewed from within the sample period Furthermore, corporate profits could be at peak levels and still cause a downward revision in expectations if still higher levels had been anticipated. It is difficult to imagine that the misspecification of the variables to be used could bias the results of a test in favor of the hypothesis when it was in fact false. The data for the tests which follow, along with the imputed source of market return taken from the relationships shown in Table 5 are summarized in table 6.

In order to show a relationship between the residual errors and the sources of market return it is only necessary to prove that the each of the two types of data was a better indicator of market return in the quarters when it was supposed to be the dominant source than in those quarters dominated by the other variable. This can be done by considering the overall success rate for the 42 significant quarters of each variable in indicating the direction of market return, then asking whether it was significantly better than this average in the quarters when it was supposed to dominate.

TABLE 6: DATA FOR QUARTERS WITH SIGNIFICANT SLOPES OF RESIDUALS VERSUS BETA

Qtr.	Year	Market Return	#Sign	(LTR)/LTR	#Sign	Corp.Prof. / Corp.Prof	#Sign	Observed Dominant Source	#Sign	Slope of Residual vs. β	Imputed Source	Sign
3	1947	0.0451	+	.0090	-	-.0102	-	Rate	-	0.0440*	Cash	+
1	1948	0.0083	+	.0209	+	.0710	+	Cash	+	0.0551*	Cash	+
1	1949	0.0285	+	-.0246	-	-.0945	-	Rate	-	-.0431*	Rate	-
2	1949	-.0660	-	0	-	-.1194	-	Cash	-	-.0795*	Cash	-
3	1949	0.1324	+	-.0672	-	.0395	+	Both		-.0527*	Rate	-
4	1949	0.1166	+	-.0135	-	-.0163	-	Rate	-	-.0296	Rate	-
2	1950	-.0050	-	.0264	+	.1957	+	Rate	+	0.0690*	Rate	+
3	1950	.1571	+	.0396	+	.2212	+	Cash	+	0.0349	Cash	+
4	1950	.1053	+	.0127	+	.0978	+	Cash	+	.1073*	Cash	+
1	1951	.0471	+	.0335	+	-.1683	-	Neither		-.0596*	Rate	-
1	1952	.0202	+	0	-	.0338	+	Both		-.0254*	Rate	-
3	1952	-.0100	-	.0383	+	0	-	Both		-.0276*	Cash	-
2	1953	-.0558	-	.0830	+	.0046	-	Both		.0261*	Rate	+
3	1953	-.0517	-	-.0383	-	-.0229	-	Cash	-	-.0596*	Cash	-
4	1953	.0374	+	-.0731	-	-.2206	-	Rate	-	-.0662*	Rate	-
3	1954	.1029	+	-.0118	-	.0609	+	Both		-.0333*	Rate	-
4	1954	.1720	+	.0278	+	.0765	+	Cash	+	.0965*	Cash	+
1	1955	.0522	+	.0734	+	.1600	+	Cash	+	.0359	Cash	+
1	1957	-.0091	-	-.0412	-	.0072	-	Cash	-	-.0319	Cash	-
3	1957	-.1039	-	.0223	+	-.0188	-	Both		-.0372*	Cash	-
4	1957	-.1036	-	-.0984	-	.0884	+	Neither		-.0898*	Cash	-
1	1958	.1238	+	-.0152	-	-.1645	-	Neither		-.0509	Rate	-

Table 6 - Continued

Qtr.	Year	Market Return	#Sign	(LTR)/LTR	#Sign	Corp. Prof. / Corp. Prof.	#Sign	Observed Dominant Source	#Sign	Slope of Residual vs. β	Imputed Source	Sign
2	1958	.0976	+	-.0165	-	.0202	+	Both		-.0266	Rate	-
3	1958	.1339	+	.1755	+	.1267	+	Cash	+	.0570*	Cash	+
2	1959	.0269	+	.0434	+	.1000	+	Cash	+	.0490	Cash	+
2	1960	.0156	+	-.0221	-	-.0380	-	Rate	-	-.0829*	Rate	-
3	1960	-.0515	-	-.0426	-	-.0827	-	Cash	-	-.0346	Cash	-
4	1960	.0519	+	.0157	+	-.0392	-	Neither		-.0905*	Rate	-
3	1961	-.0069	-	.0361	+	.0530	+	Rate	+	.0731*	Cash	-
4	1961	.0553	-	.0100	-	.0627	+	Both		-.0596*	Rate	-
1	1962	-.0085	-	-.0123	-	.0159	+	Neither		.0333	Rate	+
2	1962	-.2390	-	-.0274	-	.0065	-	Cash	-	.1217*	Rate	+
3	1962	.0166	+	.0103	+	.0194	+	Cash	+	-.0255	Rate	-
4	1962	.0907	+	-.0178	-	.0095	-	Rate	-	-.0361	Rate	-
2	1963	.0463	+	.0173	+	.0546	+	Cash	+	.0273	Cash	+
4	1963	.0057	+	.0247	+	.0417	+	Cash	+	.0661*	Cash	+
1	1964	.0664	+	.0097	-	.0602	+	Both		.0765*	Cash	+
4	1964	-.0013	+	-.0048	-	.0076	-	Rate	+	-.0356	Rate	-
1	1965	.0800	+	.0024	-	.1469	+	Both		.0362*	Cash	+
2	1965	-.0595	-	-.0024	-	.0269	+	Neither		.0428*	Rate	+
3	1965	.0955	+	.0266	+	.0131	-	Rate	-	.0696*	Cash	+
4	1965	.0955	+	.0424	+	.0647	+	Cash	+	.1414*	Cash	+
Mean for 76 Quarters		.0101				.0152				0.0		

* Significant at 1% level or better

\# Sign relative to 76 quarter mean

In terms of Table 5 this test consists of using the first two columns to determine which row an observation falls into, using only the significant observations, then determining whether the proxy variables show more than a chance association of their direction of change with that implied by the third column. Table 7 shows the results of testing the null hypothesis that the implied sources are not related to the proxy variables. The probability at the right is the probability of drawing a sample of the same or more extreme reliability equal in size to the number of quarters in which a particular variable is implied dominant by Table 5 from a total sample of either the 42 significant quarters or all quarters, which contains as many successes as that variable was correct in all included quarters. The sampling is exhaustive. Therefore, either proxy variable must do as badly in the wrong quarters as it does well in the good quarters since the second sample is left after the first is drawn. To the extent that rate changes and expectations changes are independent, the two tests are also independent. It would be possible to find an association of quarters in which rate was the implied dominant source with the rate variable and still not observe any pattern in the cash flow variable. This and the problem misspecification of the proxy variables make the results shown in Table 7 more striking. For both the long term rate and the corporate profits a lack of association must be rejected as slightly

Table 7: Probability of Chance Association Between Imputed Sources and Observed Data: Proxy Variables as Predictors of Market Returns in Significant Quarters

Sample	Number in Sample	Proxy Variable	Number of Times Proxy is Correct in Sample	Total Pool of Observations	Number of Times Proxy Correct in Total Pool	Significance Level
Cash Qtrs.	22	LTR	6	42*	20	.0065
Rate Qtrs.	20	LTR	14	42*	20	.0065
Cash Qtrs.	22	Corp.Prof.	18	42*	26	.0063
Rate Qtrs.	20	Corp.Prof.	8	42*	26	.0063
Significant Quarters	42	LTR	20	76**	38	.5911
Significant Quarters	42	Corp.Prof.	26	76**	48	.4959
Cash Qtrs.	22	LTR	6	76**	38	.0108
Rate Qtrs.	20	LTR	14	76**	38	.0334
Cash Qtrs.	22	Corp.Prof.	18	76**	48	.0268
Rate Qtrs.	20	Corp.Prof.	8	76**	48	.0136

* Significant quarters only

** All quarters 1947-1965

better than the .007 level.

Another form of test is to see whether the sign in the first column shows more than a chance association with the observed slope when the second and third columns of Table 5 are used to find the row. Here the results are limited by the smaller sample size which resulted from the fact that in 32 out of 76 quarters the source could not be identified from the proxy variables. These were selected by eliminating all quarters in which rates and corporate profits moved in opposite directions, since such movement should cause components of market movement in the same direction from both sources. For the remaining quarters the dominant source was taken to be the one which moved in the appropriate direction to cause the observed market return. The total pool from which the samples could be drawn was restricted to the 25 quarters with a significant slope of residuals versus β, and an identifiable source for market return. This limits the sample to 16 cash flow source quarters and 10 rate source quarters. Two types of test have been used to determine the association between the implied and actual signs of the slope.

The first test was to divide the sample into quarters in which a positive slope of residuals versus β was predicted and to compute the likelihood of the observed degree of association or more occuring by change. These probabilities are shown in Table 8 for samples taken from all significant quarters and for samples taken only

Table 8: Association of Actual Slopes with those Indicated by Dominant Proxy Variable

Sign of Slope Indicated	Number of times	Actual Number Which Agree	Total Number of Quarters with this Sign	Total Pool	Probability of Chance Association
+	13	10	19	42	.0072
+	13	10	13	25	.0131
-	12	9	23	42	.0917
-	12	9	12	25	.0131

Table 9: Association of Sources with Slopes

Dominant Source	Positive Slope Indicated	Negative Slope Indicated	Total Observations	Number Correct	Total Pool	Number of Positive Slopes	Number of Negative Slopes	Probability By Chance
Corp.Prof.	10	5	15	13	15	10	5	.0170
Corp.Prof.	10	5	15	13	25	13	12	.0038
Long Rate	3	7	10	6	10	3	7	.7083
Long Rate	3	7	10	6	25	13	12	.3566

from quarters in which the dominant source was determined. From the test conditional on a known source the prediction of slope is seen to be significant at the .013 level. The unconditional tests indicate a stronger association for positive slope predictions. This is almost certainly due to the fact that cash flow quarters dominated the positive predictions, whereas rate quarters dominated the negative slope quarters. As indicated by the source tests below, the cash flow association was considerably more reliable in this sample.

The second test was made by computing the likelihood of each source variable being as good a predictor of slope as it was observed to be given the distribution of slopes in the quarters it dominated,or the distribution of slopes for all significant quarters with a known source. The results of this test, tabulated in Table 9 clearly indicate that the corporate profits are related to the slope of residual vs. β, and can be used with reasonable accuracy to predict the sign of that slope. The rate variable does not fare so well. Changes in the long term rate cannot be shown to be a good indicator of the sign of the slope. It should be observed that the small size of the sample for rate quarters makes significance more difficult to achieve. Furthermore, the rate proxy variable is probably the most badly misspecified of the two. In the tests on larger sample sizes shown in Table 7 the rate variable association was strong.

Although this might be explained simply by association of the rate variable with the cash flow variable which was in turn a good predictor, this is ruled out by the data in Table 6 which indicates that the cash flow variable was a worse than average predictor in rate quarters. Therefore, the lack of significance of the rate variable in indicating the slope of residual vs. β must be viewed as a failure to provide additional confirmation for the theory rather than a failure of the theory itself.

VI. SUMMARY AND CONCLUSIONS

Working from an equilibrium model of security prices in terms of expectations and a capitalization rate, a two factor model of security returns was developed. A conversion from the pure two factor form to an expression relating security returns to returns on the market portfolio revealed that the residual errors of the single factor model should be related to the β_i's in a particular way. Association between the residual errors and the two proxy variables for the sources has been shown. This is sufficient to guarantee that over a finite time horizon the observed returns may fail to fit the capital asset pricing model simply because the mean observed return from each source deviates from the previously expected average. As a result, the validity of the link which ties the results of previous tests of the capital asset pricing model based solely on return experience to the *ex ante* equilibrium model is doubtful.

The analysis has not pursued the biases which could be introduced by changes in the risk free rate or misestimation of the β term. These would clearly cause the validity of ex post testing procedures to deteriorate further. One correction which has been used for the single factor model has been to introduce a second factor for the zero β portfolio.[9] Although this could result in nearly the same after the fact corrections as the approach described here, it does not have the same ex ante implications. Since here two distinct sources contribute to the returns of all portfolios. Furthermore these factors will be distinct from the risk free rate.[10]

In addition to its implications for testing procedures the model developed above can be used in the process of security selection and evaluation. In the development of the security β, a rationale for the short term instability but long term stability of the β terms has been given which allows for the adjustment of historical β values for current market conditions. Furthermore it has been shown that when the relationship between company cash flows and market wide cash flows is known, then only market wide parameters determine the β of the security, the capitalization rate, and hence the price. These results could bear considerable fruit in the field of security analysis and in solving problems involving required rates of return for applications from capital budgeting to regulatory rate setting if the underlining market varibles can be estimated with sufficient accuracy.

Appendix I

The Nonlinearity Model and The Observed Linearity of Returns

This appendix has been included to show why a linear relationship between returns and β will generally be observed even when there is a nonlinear component of returns as shown in figure 3. First, returns generated by a generalized quadratic function will be examined and the linearity of the observed result will be measured. This result will then be evaluated for the model of equation (4-9) by using a Taylor series approximation to the nonlinear term and substituting these equilibrium parameters into the general quadratic form. The contribution of higher than second order terms has been ignored.

A general quadratic return generating process can be written

(AI-1) $$R_i = a + b\beta_i + c\beta_i^2 + \epsilon_i$$

$$E[\epsilon, \beta_i] = 0, \; E[\epsilon, \beta_i^2] = 0$$

(AI-2) $$\mathrm{Var}[R_i] = b^2 \, \mathrm{Var}[\beta_i] + c^2 \, \mathrm{Var}[\beta_i^2] + 2bc \, \mathrm{Cov}[\beta_i, \beta_i^2] + \mathrm{Var}(\epsilon_i)$$

Suppose, however, that a linear model is estimated so that the contribution of the β^2 term must be approximated by a term in β.

(AI-3) $$\beta_i^2 = d + f\beta_i + U_i$$

Then

(AI-4) $$R_i = (a + dc) + (b + fc)\beta_i + (cu_i + \epsilon_i)$$

(AI-5) $\mathrm{Var}(R_i) = (b + fc)^2\,\mathrm{Var}(\beta_i) + c^2\,\mathrm{Var}(U_i) + \mathrm{Var}(\epsilon)$

In order to test the adequacy of the linear approximation (AI-4) in explaining the variation of R_i, the fraction of total variance of R_i which can be explained by the addition of a β^2 term to the specification of (AI-4) can be computed. To do this explicitly it is necessary to have estimates of the relative size of $\mathrm{Var}(\beta_i)$, $\mathrm{Var}(\beta_i^2)$, $\mathrm{Cov}(\beta_i, \beta_i^2)$, $\mathrm{Var}(U_i)$ and $\mathrm{Var}(\epsilon)$. The first three can be estimated simply by picking a distribution for β_i. Usually cross-section studies on portfolios have values of β_i which are well within the limits

$$.25 < \beta_i < 1.75$$

so here it will be assumed that β is uniformly distributed between those limits. This range is wider than is usually actually used and should push the results towards acceptance of the quadratic form. Under this assumption and using a sample containing 16 observations of β

$$\mathrm{Var}(\beta) \approx .2125$$

$$\mathrm{Var}(\beta^2) \approx .8783 \approx \sout{4\,\mathrm{Var}(\beta_i)}$$

$$\mathrm{Cov}(\beta, \beta^2) \approx \sout{.4116}\ .4250$$

and if U_i results from a least squares estimate

$$f = 2$$

$$d = \overline{\beta_i^2} - 2(\bar{\beta}_i)\ \sout{1.99} = -.7875$$

The fraction of variance of β^2 which is explained by the linear model, and the variance of U in terms of the variance of β are found by substitution in

$$\text{Var}(\beta^2) = f^2\ \text{Var}(\beta) + \text{Var}(U)$$

which results in

$$\text{Var}(U)/\text{Var}(\beta^2) = .0402$$

and

$$\text{Var}(U) = .1608\ \text{Var}\ (\beta) = .0354$$

Now the fraction of total variance of R_i which is left out of the explained variance when the linear model is used can be computed. This represents the theoretical limit on the increase in R^2 which would result from the addition of a quadratic term.

$$\text{(AI-6)}\quad \text{fraction left to be explained by } \beta^2 = \frac{c^2\text{Var}(U_i)}{(b+cf)^2\ \text{Var}(\beta_i) + c^2\text{Var}(U_i) + \text{Var}(\epsilon_i)}$$

substituting for the variances

$$\frac{c^2(.0354)}{(b + 2c)^2(.2125) + c^2(.0354) + \text{Var}(\epsilon_i)}$$

$$= \frac{c^2}{6.0028(b+2c)^2 + c^2 + 6.0028\ \text{Var}(\epsilon_i)}$$

$$\text{(AI-7)}\quad \approx \frac{1}{6(b/c + 2)^2 + 1 + 6\ \text{Var}(\epsilon_i)}$$

From (AI-7) it can be seen that when b and c of equation (AI-1) have the same sign, the R^2 value can be increased at most by .04 by the addition of a quadratic term. Since the sign of the curvature of the nonlinearity in the relationship between returns and β_i depends on the difference between the two factors, and the linear part of returns depends on both the sum and the difference, it will not in general be possible to determine the relationship between the sign of b and c for the quadratic approximation to (4-9) without examining it directly.

Equation (4-9) is made up of a term which is linear in β and a term which is nonlinear in β. For any sample period the values of the components of market return will be fixed and are treated as constants in the following analysis. The form to be analyzed is

$$(AI-8) \qquad R_i = \rho_n + \beta_i(\tilde{r}_m - \rho_n) + f(\beta_i)(f_2) + \epsilon_i$$

f_2 is the contribution of the difference in sources of market return and

$$f(\beta_i) = (\beta\rho_i - \beta c_i)$$

and substituting from (3-8), and (3-7) to get $f(\beta_i)$ in terms of ρ_n, ρ_m, β_n^m and β_i

$$f(\beta_i) = \frac{1}{1 - \alpha}(\beta_\rho - \beta_i)$$

(continued)

$$= \frac{-1}{1-\alpha} (\beta_i - 1 + (1 - \beta_n^m)(1 - \beta_i) \rho_n/\rho_i)$$

and from (3-6) $d\rho_i/d\beta_i = \rho_m - \rho_n$

By a Taylor series expansion around $\beta_i = 1$

(AI-) $$f(\beta_i) \approx f(1) + (\beta_i - 1) f'(1) + \frac{(\beta_i - 1)^2}{2} f''(1)$$

$$f(1) = 0$$

$$f'(1) = \frac{1}{1-\alpha} (\div (1 - \beta_n^m) \rho_n/\rho_m)$$

$$f''(1) = \frac{-1}{1-\alpha} ((\rho_m - \rho_n) \rho_n (1 - \beta_n^m)/ \rho_m^2)$$

substituting these values into (AI-9) yields

$$f(\beta_i)(\rho_n/\rho_m)(1 - \beta_n^m)/(\alpha - 1) = \frac{3\rho_m - \rho_n}{2\rho_m} - \beta_i (1 + \frac{\rho_m - \rho_n}{\rho_m}) + (\beta_i^2/2)(\frac{\rho_m - \rho_n}{\rho_m})$$

and setting $g_2 = f_2(\rho_n/\rho_m)(1 - \beta_n^m)/(\alpha - 1)$

and substituting for f_2 and $f(\beta_i)$ in (AI-8) yields

(AI-10) $$R_i = [\rho_n + (-\frac{3\rho_m - \rho_n}{2\rho_m}) g_2] + \beta_i [\tilde{r}_m - \rho_n - g_2 (1 + \frac{\rho_m - \rho_n}{\rho_m})] + \beta_i^2 [g_2 \frac{(\rho_m - \rho_n)}{2\rho_m}] + \epsilon$$

From equation (AI-7) the improvement to be gained from the addition of a β_i^2 term to the relationship between R_i and β_i will depend on the ratio b/c.

Comparing (AI-10) to (AI-1) indicates that the ratio b/c for this model is

$$b/c = [\tilde{r}_m - \rho_n - g_2(\frac{2\rho_m - \rho_n}{\rho_m})]/g_2 \ (\frac{\rho_m - \rho_n}{2\rho_m})$$

$$= \frac{2\rho_m(\tilde{r}_m - \rho_n)}{\rho_m - \rho_n}(1/g_2) - \frac{\rho_m}{\rho_m - \rho_n} - 1$$

$$= \frac{2\rho_m}{\rho_m - \rho_n}[(\tilde{r}_m - \rho_n)/g_2 - 1] - 2$$

The magnitude of $(b/c + 2)^2$ will control the variance explained by the addition of a quadratic term.

$$(b/c + 2) = \frac{2\rho_m}{\rho_m - \rho_n}[(\tilde{r}_m - \rho_n)/g_2 - 1]$$

To examine the range of this term assume that $\frac{\rho_m}{\rho_m - \rho_n} \geq 1$. Then so long as

(AI-11) $\quad 2 < (\tilde{r}_m - \rho_n)/g_2$

or

(AI-12), $\quad 0 > (\tilde{r}_m - \rho_n)/g_2$

the effects of the nonlinear term will improve the R^2 value by less .04 and therefore will not be a significant improvement unless there are a considerable number of observations on β_i and R_i. How likely $(\tilde{r}_m - \rho_n)/g_2$ is to fall within this range will depend on the probability distributions of $\tilde{r}_m$ and g_2. The frequency of occurance of insignificant amounts of curvature (taken here as an increase in R^2 of less than .04) will correspond to the probability of either condition (AI-11) or (AI-12) being met.

The two conditions are mutually exclusive. It will be assumed in the analysis which follows that $\tilde{r}_m$ and f_2 are independent. This is not strictly true unless $\alpha = 1/2$. Since the mean of f_2 and therefore g_2 is zero, it is only necessary for half the probability mass of g_2 to lie on each side of the mean to insure that condition (AI-12) will be met with probability 0.5 . This leaves only those cases to be considered for which both $(r_m - \rho_n)$ and g_2 have the same sign. If $(\tilde{r}_m - \rho_n)$ and $2g_2$ were identically distributed, then when both had the same sign condition (AI-11) would be met half the time, making the probability of observing a significant result only .25 . However the two variables are not identically distributed. $(\tilde{r}_m - \rho_n)$ has a mean greater than zero and, as shown below, a variance which is greater than that of $2g_2$. Both of these conditions increase the likelyhood of condition (AI-11) being met.

Condition (AI-11) can be written as two separate conditions.

(AI-11a) $$(\tilde{r}_m - \rho_n)/g_2 > 0$$

(AI-11b) $$|\tilde{r}_m - \rho_n| > |2g_2| \quad .$$

Recalling the definition of g_2 and making some assumptions about the size of β_n^m, ρ_n, ρ_m, and α, (AI-11b) can be rewritten in terms of f_2.

$$g_2 = f_2(\rho_n/\rho_m)(\beta_n^m - 1)(1-\alpha)$$

and if $2 \geq \beta_n^m \geq 0$

$$\rho_n/\rho_m \leq (2/3)$$

$$(1/3) \leq \alpha \leq (2/3)$$

are all satisfied then

$$|2g_2| \leq |f_2|$$

so that (AI-11) becomes

(AI-13) $$(\tilde{r}_m - \rho_n)/f_2 < 0$$

(AI-14) $$|\tilde{r}_m - \rho_n| > |f_2| \quad .$$

Now the variances of $\tilde{r}_m$ and f_2 can be compared by recalling the definitions of these variables and of α.

$$\tilde{r}_m = \rho_m + \delta\tilde{\rho}_m + (1 + {}_m)\delta\tilde{\tilde{c}}_m$$

$$f_2 = \alpha\delta\tilde{\rho}_m + (1 - \alpha)(1 + \rho_m)\delta\tilde{\tilde{c}}_m$$

and assuming independence of the factors,

$$\mathrm{Var}(\tilde{r}_m) = \mathrm{Var}(\delta\tilde{\rho}_m) + (1 + \rho_m)^2\mathrm{Var}(\delta\tilde{\tilde{c}}_m)$$

$$\mathrm{Var}(f_2) = \alpha^2\mathrm{Var}(\delta\tilde{\rho}_m) + (1-\alpha)^2(1 + \rho_m)^2\mathrm{Var}(\delta\tilde{\tilde{c}}_m)$$

substituting from the definition of α yields:

$\mathrm{Var}(f_2) =$ ~~$(\alpha^3 + (1 - \alpha)^3)\mathrm{Var}(\tilde{r}_m)$~~ $[\alpha^2(1-\alpha) + (1-\alpha)^2\alpha]\,\mathrm{Var}(\tilde{r}_m)$

$= \alpha(1-\alpha)\,\mathrm{Var}(\tilde{r}_m)$

$$(1/3) \leq \alpha \leq (2/3)$$

$$\mathrm{Var}(f_2) \leq (1/4)\mathrm{var}(\tilde{r}_m)$$

In summary, the probability of not finding a significant contribution from the addition of a quadratic term in explaining cross section returns will be greater than .5 plus the probability that

$$(\tilde{r}_m - \rho_n) > f_2$$

$$(\tilde{r}_m - \rho_n)/f_2 < 0$$

when $$E\{\tilde{r}_m - \rho_n\} = \rho_m - \rho_n$$

$$E\{f_2\} = 0$$

$$Var(f_2) < (1/4)Var(\tilde{r}_m) \quad .$$

This will total a probability of appreciably more than .75 . When there are fewer than twenty five cross sectional observations, the increase in R^2 required to make the addition of a quadratic term significant increases and it becomes increasingly unlikely that any nonlinear effect will be observed.

One means of observing the behavior of the second factor is simply to solve equation (AI-8) for the non linear term. Then in terms of observable quantities, there will be one observation of the product of the second factor and its coefficient for each portfolio whose return and β havebeen measured. If the β coefficient is estimated over the same period as returns are measured, then the linear part of the second factor's contribution will be included in the β estimate and the constant term. By using the data in this manner it should be possible to observe the factor coefficients up to a constant of proportionality and subject to the residual error. Something quite similar to this has been done by Blume and Friend {5} as shown in Appendix II. However in their estimates for the risk free rate they have

$$R_f + \frac{(\beta_{\rho i} - \beta_{ci})}{1-\beta_i} f_2$$

so that the nonlinearity of the $(\beta_\rho - \beta_c)$ term is offset by the $1/(1-\beta_i)$ term. If there were a marked curvature in their results when the effect of the $(1 - \beta_i)$ term is removed, it would indicate the sign of f_2 and hence the direction of the necessary correction in the estimate of the risk free rate. Unfortunately, the uncertainty in the estimates of β_i and the small degree of curvature available militate against using their data to confirm or deny the implied nature of the function $(\beta_{\rho i} - \beta_{ci})$.

Appendix II

Relationship to the Zero β Portfolio Model

In [5] Blume and Friend show that the return generating process which is implicit in the use of the zero β portfolio to adjust returns on risky assets from _ex post_ values to _ex ante_ expectations has the form

$$\text{(AII-1)} \quad R_i = E(R_i) + \tilde{\delta}_1 + \beta_i(\tilde{\delta}_2 - \tilde{\delta}_1) + \tilde{\epsilon}_{it}$$

In terms of the two factors $\tilde{\delta}_1$ and $(\tilde{\delta}_2 - \tilde{\delta}_1)$, the market wide factor $\tilde{\delta}_1$ has the same impact on all securities. This leads to the relationship between observable quantities:

$$\text{(AII-2)} \quad \tilde{R}_{it} = \tilde{R}_{ot} + \beta_i[\tilde{R}_{mt} - \tilde{R}_{ot}] + \tilde{\epsilon}_{it}$$

If the assumption that one factor has the same impact on all securities is removed by writing the return generating process as

$$\text{(AII-3)} \quad \tilde{R}_{it} = E[\tilde{R}_{it}] + \alpha_i \tilde{\delta}_1 + \beta_i(\tilde{\delta}_2 - \tilde{\delta}_1) + \tilde{\epsilon}_{it}$$

Then the relationship similar to (AII-2) between observables becomes

$$\text{(AII-4)} \quad \tilde{R}_{it} = \tilde{R}_{ot} + \beta_i[\tilde{R}_{mt} - \tilde{R}_{ot}] + \tilde{\epsilon}_{it} + (\alpha_i - \alpha_o)\tilde{\delta}_1 + \beta_i(\alpha_o - \alpha_m)\tilde{\delta}_i$$

Equation (AII-4) indicates that the factor $\tilde{\delta}_1$ may contribute an additional component to all portfolios other than the two used to define the factors, those with $\beta = 0$ and $\beta = 1$. Therefore, the zero β portfolio adjustment can only be complete in general in adjusting the returns of itself for sample values of the random variables generating returns. Even though there seems to be no reason to believe a priori that $\alpha_i = 1$ for all assets in equation (AIII-3), the model could still be used to make cross section estimates of R_f as done by Blume and Friend. The time series estimates, however, would be reliable for the portfolios with $\beta_i = 0$ and $\beta_i = 1$, since others could show bias due to the $\tilde{\delta}_1$ term. But these are the two cases which cannot be used; the $\beta = 0$ case because not enough data is available, and the $\beta = 1$ case because the reliability of the estimate goes to zero. The problem of drift in the error term is also present and is discussed below.

To see what bias may be inherent in the estimates of R_f, the risk free rate, under the more general case where α_i varies among assets, recall the rotation of the general two factor model given in equation (4-9).

$$\text{(AII-5)} \quad \tilde{R}_{it} = \rho_n(1 - \beta_i) + \beta_i\tilde{R}_{mt} + (\beta\rho_i - \beta c_i)(-\alpha\delta\tilde{\rho}_{mt} - (1 - \alpha)(1 + \rho_m)\delta\tilde{\bar{c}}_{mt}) + \tilde{\epsilon}_{it}$$

from this the actual return on the zero β portfolio can be written

$$\tilde{R}_{ot} = \rho_n + (\beta\rho_o - \beta c_o)(-\alpha\delta\tilde{\rho}_m - (1-\alpha)(1+\rho_m)\delta\tilde{\bar{c}}_{mt}) + \tilde{\epsilon}_{ot}$$

or

$$\text{(AII-6)} \quad \tilde{R}_{ot} = \rho_n + \mu$$

where

$$\text{(AII-7)} \qquad \mu = (\beta_{\rho o} - \beta_{co})(-\alpha\delta\tilde{\rho}_m - (1 - \alpha)(1 + \rho_m)\delta\tilde{\bar{c}}_{mt}) + \tilde{\epsilon}_{ot}$$

Three characteristics (AII-7) which are of interest here are:

1) the sample mean of μ need not be zero when secular drift in the underlying variables occurs;

2) the variance of μ will not decrease with diversification over time after the effect of $\tilde{\epsilon}_{ot}$ has been eliminated;

3) the variance of μ will be approximately $(\beta_{\rho o} - \beta_{co})^2/4\int$ times variance of R_m.

By averaging (AII-5) over time, the time series form relating observed quantities to the <u>ex ante</u> average risk free rate is found for any β level. Following Blume and Friend and dropping the t subscripts to indicate averages:

$$\tilde{R}_i = \tilde{R}_f(1-\beta_i) + \beta_i\tilde{R}_m + \tilde{\epsilon}_i + (\beta_{\rho i} - \beta_{ci})(-\alpha\delta\tilde{\rho}_m - (1-\alpha)(1+\rho_m)\delta\tilde{\bar{c}}_m)$$

over a single time series the market-wide variables $(-\alpha\delta\tilde{\rho}_m - (1-\alpha)\delta\tilde{\bar{c}}_m)$ will take on a single value for all securities and portfolios which will be written as f_2.

Solving for R_f

$$R_f = \frac{R_i - \beta_i R_m}{(1 - \beta)} - \frac{(\beta\rho_i - \beta c_i)}{(1 - \beta_i)} f_2 + \frac{\tilde{\epsilon}_i}{1-\beta} .$$

Since the sample mean value of f_2 can vary from zero there is no need for this process to provide an unbiased estimate of R_f. If f_2 is non-zero, the estimates of R_f should vary with β_i due to the variation of the coefficient of the second term with β. The arbitrary size of f_2 will make the slope of this variation indeterminant, but the very small amount of nonlinearity shown in figure 3 and discussed in Appendix I should show up in the estimates of R_f for different values of β. [redacted]

Appendix III

General Method Relating Observed Returns To Ex-ante Expectations

If returns to assets are characterized as being generated by the process described by equation (AIII-1), then this model can be used to describe the conditional expected return given the individual factor results for any time period. If the capital assed pricing model holds and all factor coefficients and sample mean values are known, then (AIII-1) can be used to relate the ex-ante capital asset pricing model parameters to the observed data. The general form given here is of particular use in analyzing less general models and approaches to the data, specifically single index models and zero β portfolio adjusted models. This theory is the basis of the current paper and underlies any fair test of the capital asset pricing model in a multiple factor world.[1]

(AIII-1)
$$\underset{n\times 1}{\underline{\tilde{R}}} = \underset{n\times 1}{\underline{\bar{R}}} + \underset{n\times k}{\underline{\underline{B}}}\ \underset{k\times 1}{\underline{\tilde{\Pi}}} + \underset{n\times 1}{\underline{\tilde{\varepsilon}}}$$

where

$\underline{\tilde{R}}$ Vector of random returns

$\underline{\bar{R}}$ Expected value of R (or sample mean)

$\underline{\tilde{\Pi}}$ Matrix of factors affecting returns of two or more securities . Expectation = 0

$\underline{\underline{B}}$ Matrix of coefficients which relate market factors to returns of individual securities

$\underline{\tilde{\varepsilon}}$ Residual errors from the linear model Expectation = 0

Now define the market portfolio with return $\tilde{R}_m$ as a particular portfolio with asset weights given by the vector $\underline{h}$. Then

(AIII-2) $$\tilde{R}_m = \underline{h}'\underline{\tilde{R}} = \underline{h}'\underline{\overline{R}} + \underline{h}'\underline{\underline{B}}\underline{\tilde{\Pi}} + \underline{h}'\tilde{\varepsilon}$$

The factors and their coefficients are unique only up to a scale factor, since a proportional increase in a row of $\tilde{\Pi}$ and decrease in the corresponding column of B would leave the model unchanged. This can be eliminated by imposing a scale constraint on the impact of each factor on some reference asset. For convenience we can constrain each factor which has an effect on the market portfolio to have as its magnetude the magnetude of its impact on the market portfolio. For those factors which do not affect the return to the market portfolio another constraint must be found. An example of this type of factor can be seen in (AIII-4). Changes in the risk free rate need not cause any change in R_m unless the two are associated outside of the relationship shown. If such outside forces cause an association such that one third of the change in the risk free rate is evidenced in the market portfolio, then the same scaling could be acomplished by either having its impact sum to its full value when projected onto a risk free portfolio, or sum to one third when projected onto the market portfolio. The zero β portfolio correction factor is an example of a factor which is presumed to have no effect on the market portfolio and is normalized to a different portfolio. In his seminal (and misleading) factor analysis King[2] required all his factors except one to be of this type. From the theory of efficient divirsification it is clear that only those factors will be of interest which can be made to project onto the return to the market portfolio. For this reason the following normalization rule will be used. Others may be added for cross-sectional studies and use in undivirsified portfolios. None is necessary unless the coefficient is to be evaluated separately from the factor value or impact of the combination.

(AIII-3) $$\underline{h}'\underline{\underline{B}} = \left(\underline{1}_j' \;\vdots\; \underline{0}\right)$$

This is a row vector with the first j elements equal to unity, and the last (k-j) elements equal to zero. This represents the fact that the first j factors sum to the deviation of return on the market portfolio about its mean, and the last (k-j) factors are completely divirsified out of the market portfolio.

The capital asset pricing model can be written

(AIII-4) $$\underline{\overline{R}} = \underline{\beta}(\overline{R}_m - R_f) + R_f \cdot \underline{1}_n$$

$\underline{\beta}$ is a vector with the generic element $\left|\mathrm{Cov}(\tilde{R}_i, \tilde{R}_m)/\mathrm{Var}(\tilde{R}_m)\right|$.

Define

$$\underline{\underline{\Sigma}} = E\{\underset{\sim}{\Pi}\underset{\sim}{\Pi}'\}$$

$$\underline{\underline{M}} = E\{\underline{\varepsilon}\underline{\varepsilon}'\}$$

Then $$\underline{\beta} = (\underline{\underline{B\Sigma B'}}\underline{h} + \underline{\underline{M}}'\underline{h})(\underline{h}'\underline{\underline{B\Sigma B'}}\underline{h} + \underline{h}'\underline{\underline{M}}'\underline{h})^{-1}$$

but if each element of h is small and the errors are reasonably independent,

$$\underline{\underline{M}}'\underline{h} \simeq \underline{h}'\underline{\underline{M}}'\underline{h} \simeq 0$$

and recalling (AIII-3) while examining the simplified relationship below

$$\underline{\beta} = \underline{\underline{B}}\left(\underline{\underline{\Sigma B}}'\underline{h}\right)(\underline{h}'\underline{\underline{B\Sigma B}}'\underline{h})^{-1}$$

an element β_i of $\underline{\beta}$ is seen to be a weighted average of the B_{ik} coefficients, where the weights correspond to the fraction of total market variance due to that factor both directly and indirectly.

Following Jensen[3] and substituting for the prior means in the market equilibrium equation:

$$\text{(AIII-5)} \qquad \underline{\bar{R}} = \underline{\tilde{R}} - \underline{\underline{B}}\underline{\tilde{\Pi}} - \underline{\tilde{\varepsilon}}$$

$$\text{(AIII-6)} \qquad \underline{\bar{R}_m} = \underline{h}'\underline{\bar{R}} = \underline{h'R} - \underline{h\underline{B}\tilde{\Pi}} - \underline{h}'\underline{\tilde{\varepsilon}}$$

Equations (AIII-5) and (AIII-6) are substituted into (AIII-4) to get an expression in terms of observable data and the realized values of the factors:

$$\underline{\tilde{R}} - \underline{\underline{B}\tilde{\Pi}} - \tilde{\varepsilon} = \underline{\beta}(\underline{h}'\underline{\tilde{R}} - \underline{h}'\underline{\underline{B}\tilde{\Pi}} - h'\tilde{\varepsilon} - R_f) + \underline{1}_n R_f$$

$$\text{(AIII-7)} \qquad \underline{\tilde{R}} = \underline{\beta}(\tilde{R}_m - R_f) + \underline{1}_n \cdot R_f + (\underline{\underline{B}} - \underline{\beta h}'\underline{\underline{B}})\underline{\underline{\tilde{\Pi}}} + \underline{\tilde{\varepsilon}} - \beta\underline{h}'\underline{\tilde{\varepsilon}}$$

In order to see what distortion is present in the use of a single index model it is helpful to define

$$\underline{\underline{\gamma}} = (\underline{\underline{B}} - \underline{\beta}(\underline{h}'\underline{\underline{B}})) \quad .$$

where γ is like the B matrix, but with β_i subtracted from each of the coefficients which sum to unity. Equation (AIII-7) can now be rewritten as

$$\text{(AIII-8)} \qquad \underline{\tilde{R}} = \underline{\beta}(\tilde{R}_m - R_f) + \underline{1}_n \cdot R_f + \underline{\underline{\gamma\tilde{\Pi}}} + (\underline{\underline{I}} - \underline{\beta h}')\underline{\tilde{\varepsilon}}$$

Equation (AIII-8) shows that in general realized returns will not depend simply on β but will involve the realized factor values of all the factors that do not divirsify out in the group of securities being measured.[4] This dependence will occur whenever all the factor coefficients are not identically equal to β_i for the security. This general model can be used to specify different return generating processes for different classes of assets in a manner which is consistent with the capital asset pricing model. Returns for a given β range could vary widely because of the different factors and weightings used to adjust the ex-post values to ex-ante means without contradicting the equilibrium relationship.

Appendix III Footnotes

1. This problem is considered in the context of the preceeding paper and the references which follow apply here as well.

2. In the light of what now seems obvious, King's study told a great deal about what no one should care about (divirsifiable risk), but kept further attention from being focused on multi-factor models. King, B.F. "Market and Industry Factors in Stock Price Behavior", Journal of Business, (January 1966)

3. Jensen, Michael C. "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios," Journal of Business (april 1969)

4. In a paper based on an earlier draft of this model Brennan tested the specification of (AIII-8) and found that a total of three factors were sufficient to explain virtually all the communality of returns for securities grouped by β. M.J.Brennan. "Capital Asset Pricing and the Structure of Security Returns," Unpublished mimeo (May, 1971)

FOOTNOTES

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[1]Sharpe [1961].

[2]Considerable evidence has been amassed to indicate that ex post returns do not in fact fit the capital asset pricing model directly. See, for example Douglas [1969], Miller and Scholes [1971], Black, Jensen and Scholes [1971], and Friend and Blume [1970].

[3]The econometric problems of testing the capital asset pricing model have been given extensive investigation by Miller and Scholes [1971], and also by

[4]One example of this type of long term secular change would be the increased stability (reduced risk) of the overall economy as the role of the government in stabilizing the economy has increased. This has involved learning throughout the past twenty five years and has clearly led in a direction which would not have been anticipated in 1947.

[5]The only method appears to be through the use of a market equilibrium model which would give expected returns as a function of other known variables. However, this will require some verification of the equilibrium model to be used.

[6]This type of model has been extended to allow for different time patterns of earnings expectations and time variant capitalization rates without drastically altering the results.

[7]One strong inducement to negative covariance between risky and riskfree rates comes from changes in the perceived risk level of national wealth. Such changes will cause a shift either from debt to equity holdings or visa versa and therefore a realignment of the relative rates which moves them in opposite directions, to a new equilibrium level. This effect is probably much more important to realized values of β_n^m than to expected values.

[8] This analysis is based on an assumption of no growth in earnings per dollar invested. If growth in output of current capital is considered then it will no longer be true that the relative weights on βo_i and βc_i will be the same for all companies. Thus βc_i alone will not be sufficient to determine β_i.

[9] The zero β portfolio and the adjustment of realized returns is developed in Black [1970], and applied to observed data in Black, Jensen and Scholes [1971], and by Blume and Friend [1971].

[10] See Appendix II for a further development.

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