Portfolio and Capital Market Theory with Arbitrary Preferences and Distributions -- The General Validity of the Mean-Variance Approach in Large Markets

by

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The concept that in equilibrium the anticipated return on a risky asset must be sufficient to compensate the investor for risk has long been a part of traditional market lore. Yet the notion of a risk premium was only first made theoretically rigorous in the work of Sharpe [1964] and Lintner [1965]. The central results of the mean-variance capital asset pricing models can be summarized in the fundamental price equation

$$E_i - \rho = \beta_i \ (E_m - \rho),$$

where $E_i - \rho$ is the risk premium, the difference between the expected return on the i^{th} asset, E_i, and ρ, the risk free rate of return, $E_m - \rho$ is the premium offered by the market portfolio, E_m, over ρ and β_i, the "beta coefficient" is proportional to the covariance between the i^{th} asset and the market portfolio. The risk premium is, thus, proportional to the degree to which the asset return and the market return move together. A security which moves inversely to the market may be so valuable in diversified portfolios that it earns a negative premium, i.e., its return is actually below the risk free return. Indeed, the major contribution of the mean-variance capital asset pricing model may well be that it formalizes the dependence of the risk premium on the relationship between the asset and the market. One of the purposes of this paper is to demonstrate that this result holds under much more general conditions than are required to develop the mean-variance pricing model. In particular, if the price equation or something similar did not hold at least approximately a type of an arbitrage situation would emerge that would enable investors to earn arbitrarily large returns while assuming only modest risk and this would be incompatible with market equilibrium.

This is an important result; in recent years a considerable effort has been devoted to finding the exact conditions under which the mean-variance approach to portfolio theory is valid and, more generally, under which the separation theorem holds. The outcome of this effort has been somewhat disappointing in that it has revealed that the mean-variance rules remain valid only under quite restrictive assumptions. Either preferences must be expressible by the objectionable quadratic von Neumann-Morgenstern utility function or investments must essentially be normally distributed. The separation theorem has proved only moderately more robust: Cass and Stiglitz [1970] have succeeded in completely characterizing the class of utility functions which permit separation in arbitrary risky environments and Ross [1971; 2] has solved the dual problem of finding the classes of distributions that allow separation for arbitrary preferences. Both classes are rich but fundamentally limited, and these results have cast doubts on the attempts to characterize portfolio choice by two dimensions and, in particular, have weakened the generality of the mean-variance approach.

Yet the mean-variance analysis has both a simplicity and a heurism to recommend it, and there does not appear to be any ready substitute. If exact validity is too much to hope for, perhaps there is an approximate sense in which the results remain true. It has been suggested that the large sample laws of probability may imply that a world with many securities behaves like a mean-variance world. If distributions possess moments above the first, then by the Central Limit Theorem portfolios that are appropriate sums of independent investments will be approximately normally distributed. The more powerful limit laws of large numbers imply that with sufficient diversification some types of risk may actually disappear in large portfolios. The intent of this paper is to rigorously explore these issues and to find concrete bounds on the errors involved in using the mean-variance approach where it is not strictly justified.

That rigor is required and not simply an opportunity to exercise pedantry in matters where intuition is self-evident can be quickly made clear. If, for example, the set of feasible investments is such that for some preference towards risk the brunt of the portfolio is placed in one, two or a small number of securities, then no limit theorems can be appealed to and, in general, the mean-variance approach will not even be approximately valid. It seems clear, then, the conditions under which all optimal portfolios are greatly diversified are roughly the sort of conditions which will justify the mean-variance approach as an approximation.[1]

There are, then, two distinct points made in this paper. The first is that the basic equilibrium condition of the capital asset pricing model is essentially an arbitrage relation and may, therefore, be expected to be quite robust. The second is that this result can be derived formally from the general validity of the mean-variance approach in large markets and this latter point is developed rigorously.

Section I considers the individual portfolio problem and some of the relevant issues in a fairly general setting obtaining some interesting and quite general theorems. Section II specializes the feasible set of investments to what is generally called the class of one-factor market models. Section III examines the implications of the previous results for the separation theorem and for their applications to the problem of equilibrium in capital markets. The resulting arbitrage arguments are developed here and the reader interested only in this point might skip the previous sections and begin here. Section IV generalizes the results obtained in the previous three sections, considering, in particular, the m-factor case, and Section V briefly summarizes the findings. An appendix is included to provide proofs of some of the results of a purely mathematical nature needed to support the arguments in the text but somewhat tangential in nature.

Section I

The problem considered in this Section is the familiar (static) portfolio problem. The individual is assumed to possess a concave (risk-averse) von Neumann-Morgenstern utility function $U(\cdot)$, whose expected value he maximizes subject to choosing an investment in the feasible set of options and subject to his budget constraint. Formally, the investor seeks

$$\text{(A)} \qquad \text{s.t.} \qquad \begin{aligned} \max_{\alpha} V(\alpha) &\equiv \underset{\tilde{x}}{E}\{U(\alpha\tilde{x})\} \\ \alpha e &= 1 , \end{aligned}$$

where $\tilde{x} \equiv \langle\tilde{x}_1,\ldots,\tilde{x}_n\rangle$ is the vector of random (gross) returns per unit of investment in securities $1,\ldots,n$; $\alpha \equiv \langle\alpha_1,\ldots,\alpha_n\rangle$ is the vector of proportions of wealth placed in the investments and $e = \langle 1,\ldots,1\rangle$. For convenience we have normalized initial wealth to be unity.[2] In Section III we will consider the case where one (or some linear combination) of the $\tilde{x}_j$'s is riskless, but for the moment it is easiest to think of the $\tilde{x}_j$'s as being random. In general, we will not be concerned with problem (A) but, rather, with a restricted problem with the additional constraint that $\alpha E = m$, where E is an n-vector and $E_j \equiv E\{\tilde{x}_j\}$. By varying m we can see how the optimal portfolio α^o changes as a function of the mean return on the portfolio. The unrestricted optimization problem then reduces to the best choice of m or, more generally, the optimal tradeoff between return and risk. It is precisely this tradeoff with which no approximation method could successfully come to grips and from which the restricted problem abstracts. Indeed, the mean-variance approach itself is designed only to elucidate the efficient frontier of the feasible investment set from which the optimal choice is drawn.

Associated with the restricted problem is the associated quadratic problem (AQP):

$$\text{(AQP)} \qquad \text{s.t.} \qquad \min \operatorname{Var}\{\alpha x\}$$
$$\alpha e = 1$$
$$\text{and}$$
$$\alpha E = m ,$$

and the optimal solution to this problem will be denoted by α^q .

There are two distinct types of approximation questions which can now be asked. First, we can inquire what penalty is exacted by using the quadratic approximation. Specifically, what is

$$(E\{U(\alpha^o x)\} - E\{U(\alpha^q x)\})$$

and how does it behave with an increase in the number of securities n ? The second and more subtle question is whether and in what, if any, sense the actual portfolios α^o and α^q become similar. If $\|\cdot\|$ denotes a norm, we will want to know what happens to $\|\alpha^o - \alpha^q\|$ as n grows large. The uniform (or sup) norm will be used to compare the two vectors α^o and α^q , i.e., $\|\alpha^o - \alpha^q\| \equiv \sup |\alpha_j^o - \alpha_j^q|$.[3]

In this section we will concentrate on approximation results of the first type. Letting V_n denote the covariance matrix of $\langle x_1,\ldots,x_n\rangle$,[4] the first order conditions for a solution to the AQP are simply

$$\alpha^q = \lambda V_n^{-1}E + \theta V_n^{-1}e , \qquad (1)$$

where λ and θ are Lagrange multipliers obtained from the constraints as

$$\begin{bmatrix} \lambda \\ \theta \end{bmatrix} = \Delta^{-1} \begin{bmatrix} E'V_n^{-1}e - m(e'V_n^{-1}e) \\ m(e'V_n^{-1}E) - E'V_n^{-1}E \end{bmatrix} \tag{2}$$

with

$$\Delta \equiv (e'V_n^{-1}E)^2 - (E'V_n^{-1}E)(e'V_n^{-1}e) .$$[5]

An important and familiar case occurs when the $\tilde{x}_j$ are mutually independent (or have zero covariance). The covariance matrix V_n is then diagonal and (1) becomes

$$\alpha_i^q = \sigma_i^{-2}(\lambda E_j + \theta) , \tag{3}$$

where

$$\begin{bmatrix} \lambda \\ \theta \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \sum_j (E_j - m)\sigma_j^{-2} \\ \sum_j (m - E_j)E_j\sigma_j^{-2} \end{bmatrix} ,$$

$$\Delta \equiv (\sum_j E_j\sigma_j^{-2})^2 - (\sum_j E_j^2\sigma_j^{-2})(\sum_j \sigma_j^{-2}) ,$$

and

$$\sigma_i^2 \equiv V_{ii} \text{ , the variance of } \tilde{x}_i .$$

It is convenient to write the return on the quadratic portfolio as

$$\tilde{R}^q \equiv \alpha^q\tilde{x} = \alpha^q E + \alpha^q y$$

$$= m + \alpha^q\tilde{y} ,$$

with $\tilde{y} \equiv \tilde{x} - E$ and $E\{\tilde{y}\} = E\{\tilde{x} - E\} = 0$. This separates the return component m from the risk $\alpha^q\tilde{y}$. On the assumption that the $\langle\tilde{x}_j\rangle$ and, therefore, the $\langle\tilde{y}_j\rangle$ are independent, we can now use the law of large numbers to demonstrate that the risk $\alpha^q\tilde{y} \to 0$ almost surely (a.s.).[6] The complete conditions sufficient for convergence are spelled out in detail in the first theorem.

Letting

$$A_n \equiv \sum_{j=1}^{n} \sigma_j^{-2} \tag{4)(a}$$

and

$$B_n \equiv A_n^{-1} \sum_{j=1}^{n} [E_j - A_n^{-1} \sum_{j=1}^{n} E_i \sigma_i^{-2}]^2 \sigma_j^{-2} \tag{4)(b}$$

we have the following result.

<u>Theorem I</u>: If $\langle\sigma_j\rangle$ and $\langle E_j\rangle$ are uniformly bounded, $\langle B_n\rangle$ is uniformly bounded away from zero, and

$$E\{\tilde{y}_n \mid y_1,\ldots,y_{n-1}\} = 0 , \tag{A(1)}$$

then

$$\alpha^q\tilde{y} \to 0 , \text{ (a.s.)} .$$

<u>Proof</u>: See Appendix

The need for the bounds in the theorem is best illustrated by means of examples. The need to restrict $\langle\sigma_j^2\rangle$ and $\langle E_j\rangle$ is obvious. If E_j were to diverge to, say, $+\infty$ sufficiently rapidly, then a subsequence that diverged monotonically could be chosen. The portfolio contributions of the members of this subsequence α_i^q need not go to zero as $0(\frac{1}{n})$,[7] and the limit conditions would not be satisfied. Intuitively, the increasing mean makes the later members of this subsequence so desirable that non-negligible proportions of wealth are put in them. Similarly, if σ_j^2 were to diverge rapidly, the risk-averse character of the mean-variance formalism might place non-negligible proportions of the portfolio in early securities.[8]

The restriction on $\langle B_n \rangle$, however, is somewhat less obvious than the others. Essentially, it requires that there be sufficient variability in the $\langle E_j \rangle$ sequence. Notice that B_n is simply a sample variance of the $\langle E_j \rangle$ sequence using for a sample distribution of $\langle \sigma_1^{-2}/A_n, \ldots, \sigma_n^{-2}/A_n \rangle$. A simple example will illustrate the need for the restriction on $\langle B_n \rangle$. Suppose that there are three types of (independent) securities with identical variances but distinct means (e_1, e_2, e), and suppose we let the population grow by adding n independent securities of type e, keeping one each of types e_1 and e_2. From A(3) for $j = 1,2$,

$$\alpha_j^q = \Delta^{-1}\{([e - m]\, e_j + em - e^2)\, n + e_j(e_1 + e_2 - 2m) + (e_1 + e_2)\, m - (e_1^2 + e_2^2)\},$$

and for $j > 2$,

$$\alpha_j^q = \Delta^{-1}\{e(e_1 + e_2 - 2m) + (e_1 + e_2)\, m - (e_1^2 + e_2^2)\},$$

where

$$\Delta = [2e(e_1 + e_2) - (e_1^2 + e_2^2) - 2e^2]\, n - (e_1^2 + e_2^2).$$

Now, as $n \to \infty$, for $j = 1,2$,

$$\alpha_j^q \to \frac{(m - e)(e - e_j)}{2e(e_1 + e_2 - e) - (e_1^2 + e_2^2)} ,$$

and for $j > 2$,

$$\alpha_j^q \to 0(\frac{1}{n}) .$$

Thus, while the risk involved from the third type of security will be completely diversified away, that from the first two will remain and

$$\alpha^q\tilde{y} \to [\frac{(m - e)}{2m(e_1 + e_2 - 3) - (e_1^2 + e_2^2)}] [(e - e_1)\tilde{y}_1 + (e - e_2)\tilde{y}_2] .$$ [9]

From Theorem 1 it is a short step to the result that

<u>Theorem 2</u>:

$$E\{U(\alpha^o\tilde{x})\} - E\{U(\alpha^q\tilde{x})\} \to 0 .$$

<u>Proof</u>:

We need only to make use of the principle of stochastic dominance which asserts that given any two risky choices $\tilde{U}$ and $\tilde{V}$, $\tilde{U}$ will be preferred to $\tilde{V}$ by every risk-averse investor if and only if

$$\tilde{V} \sim \tilde{U} - \tilde{Z} + \tilde{\epsilon} ,$$

where " $\sim$ " denotes that $\tilde{V}$ and $\tilde{U} - \tilde{Z} + \tilde{\epsilon}$ are identically distributed, $\tilde{Z}$ is a non-negative random variable, and $\tilde{\epsilon}$ is a conditionally independent error term; i.e., $E\{\tilde{\epsilon} \mid \tilde{U} - \tilde{Z}\} = 0$ (see Ross [1971; 1]). Since we have shown that

$$\tilde{R}^q \equiv m + \alpha^q\tilde{y} \to m , \quad (a.s.)$$

and since

$$\tilde{R}^o \equiv m + \alpha^o\tilde{y}$$

is inferior to m by the principle of stochastic dominance, it must follow that $\tilde{R}^o \to m$, (a.s.).

Q.E.D.

Notice, though, that neither Theorem 1 nor Theorem 2 is sufficient to imply that the two portfolios α^o and α^q approach each other. It is fairly easy to see that both $\|\alpha^o\| \to 0$ and $\|\alpha^q\| \to 0$, and, therefore, $\|\alpha^o - \alpha^q\| \to 0$, but this is not very interesting since both vectors are approaching the zero vector. What would be interesting to show is that for all i , (α_i^q/α_i^o) was bounded, or even better, $(\alpha_i^q/\alpha_i^o) \to 1$. This second, stronger approximation result will be the subject of the next section as will the generalizations of Theorems 1 and 2 to the case where $\langle\tilde{x}_1,\ldots,\tilde{x}_n\rangle$ are dependent.

Section II

To begin with we will assume that the returns $\langle\tilde{x}_1,\ldots,\tilde{x}_n\rangle$ are generated by the following simple one-factor market model:

$$\tilde{x}_j = E_j + \beta_j\tilde{\delta} + \tilde{\epsilon}_j \; ; \qquad j=1,\ldots,n , \tag{5}$$

where E_j and β_j are constants (the latter is generally referred to as the volatility of $\tilde{x}_j$), $\tilde{\delta}$ is a random market factor common to all securities with $E\{\delta\} = 0$ and $Var\{\tilde{\delta}\} = \sigma^2$, and $\tilde{\epsilon}_j$ is an error term, mean zero and conditionally independent of $\tilde{\delta}$ and the other error terms.[10] The market factor $\tilde{\delta}$ is variously interpreted as (anticipated) GNP or some index of overall economic performance. The restrained portfolio problem is now

$$\max_{\alpha} E\{U(\alpha x)\}$$

s.t.

$$\alpha e = 1 ,$$

and

$$\alpha E = m ,$$

and the AQP is

$$\min \operatorname{Var}(\alpha x) = \alpha V_n \alpha + (\alpha \beta)^2 \sigma^2$$

$$= \alpha [V_n + \sigma^2 \beta \beta'] \alpha$$

$$\equiv \alpha M \alpha$$

s.t.

$$\alpha e = 1$$

and

$$\alpha E = m .$$ [11]

The solution to this AQP is identical to that given in (1) with V_n replaced by M .

The central theorem of this section asserts that the second type of approximation $\alpha^q \approx \alpha^o$ is valid with the single-factor model. We will build to the theorem with two preliminary lemmas.

Lemma 1:

If we assume that $\langle\beta_j\rangle$, $\langle E_j\rangle$, and $\langle\sigma_j\rangle$ are bounded as in Theorem 1, that $e'V^{-1}E$, $e'V^{-1}\beta$, $\beta'V^{-1}E$, $\beta'V^{-1}\beta$, and $e'V^{-1}e$ are of the same order and that there exist positive constants a and b such that

$$(e'V^{-1}e)^{-1}\{\beta'V^{-1}\beta)\ (e'V^{-1}e) - (e'V^{-1}\beta)^2\} \geq a(e'V^{-1}e)\ , \qquad (6)\ (a)$$

and

$$B \equiv (e'M^{-1}e)^{-2}\ \{(E'M^{-1}E)\ (e'M^{-1}e) - (e'M^{-1}E)^2\} \geq b\ , \qquad (6)\ (b)$$

then

$$\alpha^q \to 0(\tfrac{1}{n}) \to 0$$

and

$$\alpha^q\beta \to 0(\tfrac{1}{n}) \to 0\ .$$

Proof: See Appendix

An important corollary of Lemma 1 is the rather startling conclusion that as n increases the market model implies the disappearance of all risk, including systematic risk.

Corollary 1: Under the conditions of Lemma 1,

$$\tilde{R}^q \equiv \alpha^q\tilde{x} \to m\ ,\ \text{a.s.}$$

Proof: From (5),

$$\begin{aligned} \tilde{R}^q &\equiv \alpha^q\tilde{x} \\ &= m + (\alpha^q\beta)\tilde{\delta} + \alpha^q\tilde{\epsilon} \end{aligned}$$

and since

$$\alpha^q\beta \to 0(\tfrac{1}{n}) \to 0$$

and

$$\alpha^q \to 0(\frac{1}{n}) \to 0 \ ,$$

the law of large numbers implies that the risk component term

$$(\alpha^q \beta)\tilde{\delta} + \alpha^q \tilde{\epsilon} \to 0 \ , \ (\text{a.s.}) \ .$$

The proof is now identical to that of Theorem 2.

<u>Q.E.D.</u>

The boundedness conditions in the statement of Lemma 1 as in that of Theorem 1 are designed to prevent either the asymptotic appearance of a riskless asset or, more generally, the asymptotic concentration of securities. In the market model, however, in the absence of these conditions there is an additional source of risk, systematic risk, which need not be diversified away. Suppose, for example, that all securities have the means $a + r\beta_i$ save for the first which has a mean $E_1 + r\beta_1$, $E_1 \neq a$. This case can be shown to violate (6) (b).[12] More directly, though,

$$\begin{aligned}
\tilde{R} \equiv \alpha\tilde{x} &= m + (\alpha\beta)\tilde{\delta} + \alpha\tilde{\epsilon} \\
&= m + \frac{1}{r}[m - a + \alpha_1(a - E_1)]\tilde{\delta} + \alpha\tilde{\epsilon} \\
&\to m + \frac{1}{r}[m - a + \alpha_1(a - E_1)]\tilde{\delta} + \alpha_1\tilde{\epsilon}_1 \ , \ (\text{a.s.}),
\end{aligned}$$

if α_j $j \neq 1$, is of $0(\frac{1}{n})$. There remains, however, a fundamental choice between the two types of risk $\tilde{\delta}$ and $\tilde{\epsilon}_1$, whenever $m \neq a$. To set $\alpha_1 \to 0(\frac{1}{n})$ would diversify away all independent variation, but the market risk would approach $[\frac{m - a}{r}]\tilde{\delta}$. On the other hand, if the market risk is set equal to zero, the systematic risk from the first asset will be non-negligible and will be given by $[\frac{m - a}{E_1 - a}]\tilde{\epsilon}_1$. This is a type of case that the requirement (6) (b) that E display sufficient variation as measured by M rules out.

A somewhat more interesting case where all securities have mean $E_i = \rho + r\beta_i$ is also ruled out by the conditions of Lemma 1. This is the security line criterion for capital market equilibrium. In this case, though, the tradeoff between the two types of risk is also eliminated since the portfolio must be subjected to a systematic risk $[\frac{m - \rho}{r}]\tilde{\delta}$ to attain a mean return of m . The unsystematic or independent risk, however, can still be diversified away. Stating this formally, we have the following corollary.

<u>Corollary 2</u>: If the market model is restricted to

$$\tilde{x}_j = E_j + \beta_j\tilde{\delta} + \tilde{\epsilon}_j$$

where

$$E_j = \rho + r\beta_j$$

and the conditions of Lemma 1 other than (6) (b) are satisfied, then

$$\alpha^q \to 0(\frac{1}{n}) \to 0 \ .$$

This implies that all of the unsystematic risk will be diversified away and

$$\tilde{R}^q \equiv \alpha^q\tilde{x} \to m + [\frac{m - e}{r}]\tilde{\delta} , \quad (a.s.) \ .$$

Proof: Identical to proof of Theorem 2. Q.E.D.

In summary, though, outside of this special equilibrium condition, if all of Theorem 2's conditions are satisfied, then $\tilde{R} \equiv \alpha^q\tilde{x} \to m$ and all risk including the so-called systematic component associated with the β weights may be diversified away. This is in contrast to the statement by Sharpe that ". . .the portfolio risk due to uncertainty about the level of the index is no smaller than that associated with the average security" (Sharpe [1970]). Sharpe was referring to a portfolio where $\alpha_j = 1/n$, but this illustrates the dangers involved in implying that the market risk as represented by $\tilde{\delta}$ is necessarily unavoidable. This will be true only when the capital market is in equilibrium.

Before proving the central theorem, we need one additional intermediate result.

Lemma 2: The optimal solution α^o and $\alpha^o\beta$ are bounded of order $1/n^{\frac{1}{2}}$, i.e.,

$$\|n^{\frac{1}{2}}\alpha^o\|$$

and

$$|n^{\frac{1}{2}}\alpha^o\beta| \quad \text{are bounded.}$$

Proof: See Appendix

We can now prove our main theorem.

Theorem 3: Under the assumptions of Lemma 1 on the market model,

$$\tilde{x}_j = E_j + \beta_j\tilde{\delta} + \tilde{\varepsilon}_j ,$$

the optimal solution α^o to the portfolio problem

$$\max E\{U(\alpha'\tilde{x})\}$$

s.t.

$$\alpha' e = 1$$

and

$$E\{\alpha'\tilde{x}\} = \alpha' E = m$$

converges to the solution α^q to the associated quadratic problem in the sense that

$$\|n\alpha^o - n\alpha^q\| \to 0(\frac{1}{n}) \to 0 ,$$

as $n \to \infty$.

<u>Proof</u>: The first order conditions for a solution to the exact optimal problem are given by

$$E\{U'(\alpha^o\tilde{x})\ (\beta\tilde{\delta} + \tilde{\epsilon})\} = \lambda^o E + \theta^o e , \tag{7)(a}$$

$$\alpha^o e = 1 ,$$

and (7)(b)

$$\alpha^o E = m .$$

(Note that since $\alpha^o\tilde{x} = m + (\alpha^o\beta)\tilde{\delta} + \alpha^o\tilde{\epsilon}$, we have subsumed the m term in λ^o). Expanding (7) (a) in a Taylor's series about m yields

$$\begin{aligned} E\{U'(\alpha^o\tilde{x})(\beta\tilde{\delta} + \tilde{\epsilon})\} &= E\{U'(m + (\alpha^o\beta)\tilde{\delta} + \alpha^o\tilde{\epsilon})(\beta\tilde{\delta} + \tilde{\epsilon})\} \\ &= E\{U'(m)(\beta\tilde{\delta} + \tilde{\epsilon})\} \end{aligned} \tag{8}$$

$$+ E\{U''(m)((\alpha^{o}\beta)\tilde{\delta} + \alpha^{o}\tilde{\epsilon})(\beta\tilde{\delta} + \tilde{\epsilon})\}$$

$$+ \tfrac{1}{2}E\{U'''(m + \tilde{y})((\alpha^{o}\beta)\tilde{\delta} + \alpha^{o}\tilde{\epsilon})^{2}(\beta\tilde{\delta} + \tilde{\epsilon})\} ,$$

where $\tilde{y}\epsilon[0,(\alpha^{o}\beta)\tilde{\delta} + \alpha^{o}\tilde{\epsilon}]$. Letting $c \equiv U''(m)$ and

$$\gamma \equiv \tfrac{1}{2}E\{U'''(m + \tilde{y})((\alpha^{o}\beta)\tilde{\delta} + \alpha^{o}\tilde{\epsilon})^{2}(\beta\tilde{\delta} + \tilde{\epsilon})\}$$

allows us to rewrite (8) as

$$cM\alpha^{o} + \gamma = \lambda^{o}E + \theta^{o}e . \qquad (9)$$

Lemma 2 asserts that $\alpha^{o} \to 0(1/n^{\frac{1}{2}})$ and $\alpha^{o}\beta \to 0(1/n^{\frac{1}{2}})$ which, together with Lemma 2A in the appendix, can be shown to imply that $\gamma \to 0(1/n)$. This result will allow us to obtain the sharper result that $\gamma \to 0(1/n^{2})$. First, we must substitute (9) into the constraints (7) (b) and solve for (λ^{o},θ^{o}) . Some algebra yields

$$\begin{bmatrix} \lambda^{o} \\ \theta^{o} \end{bmatrix} = c \begin{bmatrix} \lambda^{q} \\ \theta^{q} \end{bmatrix} + A^{-1} \begin{bmatrix} e'M^{-1}\gamma \\ E'M^{-1}\gamma \end{bmatrix} \equiv c \begin{bmatrix} \lambda^{q} \\ \theta^{q} \end{bmatrix} + \begin{bmatrix} \gamma_{\lambda} \\ \gamma_{\theta} \end{bmatrix}$$

where

$$A \equiv \begin{bmatrix} e'M^{-1}E & e'M^{-1}e \\ E'M^{-1}E & e'M^{-1}E \end{bmatrix} ,$$

and by (9) we obtain

$$
\begin{aligned}
c\alpha^{o} &= \lambda^{o}M^{-1}E + \theta^{o}M^{-1}e - M^{-1}\gamma \\
&= c\lambda^{q}M^{-1}E + c\theta^{q}M^{-1}e + \gamma_{\lambda}M^{-1}E + \gamma_{\theta}M^{-1}e - M^{-1}\gamma \qquad (10) \\
&= c\alpha^{q} + [\gamma_{\lambda}M^{-1}E - \gamma_{\theta}M^{-1}e - M^{-1}\gamma] .
\end{aligned}
$$

From Lemma 1, $\alpha^{q} \to 0(\frac{1}{n})$, and from the boundedness assumptions together with the observation that $\gamma \to 0(\frac{1}{n})$, the bracketed term can also be shown to be $\to 0(\frac{1}{n})$. But this implies that $\alpha^{o} \to 0(\frac{1}{n})$, and since $\alpha^{q}\beta \to 0(\frac{1}{n})$, we can also show that $\alpha^{o}\beta \to 0(\frac{1}{n})$. Hence, by the second application of Lemma 2A in the appendix we now have $\gamma \to 0(1/n^{2})$. At this stage the proof is immediate. The bracketed term of (10) $\to 0(1/n^{2})$ and, therefore,

$$\|\alpha^{o} - \alpha^{q}\| \to 0(\frac{1}{n^{2}}) .$$

Hence,

$$\|n\alpha^{o} - n\alpha^{q}\| \to n0(\frac{1}{n^{2}}) \to 0(\frac{1}{n}) \to 0 .$$

Q.E.D.

The sense of approximation in Theorem 3 is quite important. As we noted earlier, α^{o} and α^{q} are becoming increasingly diversified and approach a zero holding in each security as n grows large. It is trivially true, then, that for large n, $\alpha_{i}^{o} \approx \alpha_{i}^{q}$. Theorem 4 asserts that $n\alpha_{i}^{o} \approx n\alpha_{i}^{q}$ and that the difference is of order $\frac{1}{n}$.[13] Thus, in a fundamental sense the optimal portfolio for any risk averse investor will be closely approximated by the minimum variance portfolio. Theorem 3 can also be easily extended to the situation covered in Corollary 2.

Theorem 4: If the market model is restricted so that $E_i = \rho + r\beta_i$ and the conditions of Lemma 2 other than (6) (b) are satisfied, then

$$\|n\alpha^q - n\alpha^o\| \to 0(\frac{1}{n}) \to 0 \ .$$

Proof: See proof of Theorem 3. Q.E.D.

In addition, both Theorems 3 and 4 can be generalized so as to allow for the existence of a risk free asset (or portfolio).

Theorem 5: If there is a riskless asset under the conditions of either Theorem 3 or Theorem 4,

$$\|n\alpha^q - n\alpha^o\| \to 0(\frac{1}{n}) \to 0 \ ,$$

where α^o and α^q now include the holdings of the riskless asset.

Proof: A simple modification of the proof of Theorem 3. Q.E.D.

In particular, then, since the separation theorem holds for the AQP, i.e., since the risky portfolio is independent of the mean return m in the quadratic problem, this same independence will be approximately true for the exact optimal problem. Some care, however, has to be exercised in the interpretation of this result. The above theorems demonstrated convergence for a particular utility function and a particular mean constraint. If the separation theorem holds, then, in effect, we can drop the mean constraint and, hopefully, still obtain an approximation result that holds for the risky portfolio. But, it should be clear that we cannot drop the mean constraint and simultaneously retain the variation conditions of Lemma 1 and hope to obtain an approximation theorem. As n gets large, in the presence of a riskless asset, it is possible to construct a portfolio with mean m whose risk approaches zero.

If, say, $m > \rho$, the investor could borrow at the riskless rate (short the riskless asset) and invest in the risky portfolio to obtain returns sufficiently high as to make the risk-return tradeoff (between the systematic risk associated with higher returns and nonsystematic risk) again meaningful regardless of the choice of n . Even if there is no risk free asset, this same difficulty arises since the investor can construct two portfolios of very low risk, one of which has a greater mean than the other. If we drop the constraint on the mean, then we must also drop the variation conditions of Lemma 1.

This point will be taken up below when we discuss capital market equilibrium. For the moment we will content ourselves with the following results.

Theorem 6: If there is a riskless asset with a return of ρ and if $E_i = \rho + r\beta_i$, then under the conditions of Lemma 1 other than (6) (b),

$$\|n\alpha^q - n\alpha^o\| \to 0(\tfrac{1}{n}) \to 0 ,$$

where α^q is the portfolio of risky assets alone (obtained from separation), and α^o is the portfolio of risky assets chosen by an individual with a particular utility function $U(\cdot)$. In addition,

$$|\alpha_o^o - \alpha_o^q| \to 0(\tfrac{1}{n}) \to 0 ,$$

where α_o^q is the proportion of wealth placed in the quadratic portfolio if that is the only risky choice available, and α_o^o is the porportion placed in the optimal portfolio.

Proof: Notice first that it was necessary to put $E_i = \rho + r\beta_i$ rather than $E_i = a + r\beta_i$, $a \neq \rho$ to prevent the construction of a nearly riskless portfolio with mean $a \neq \rho$. The proof of the theorem follows from an examination of the return. For any choice of (a_o, α) , where α_o is the portion of the portfolio invested in risky assets and α with $\alpha e = 1$ is the risky portfolio, the random return,

$$\tilde{R} \equiv (1-\alpha_o)\rho + \alpha_o(\tilde{\alpha x})$$

$$= (1-\alpha_o)\rho + \alpha_o[\alpha E + (\alpha\beta)\tilde{\delta} + \alpha\tilde{\epsilon}]$$

$$= [(1-\alpha_o)\rho + \alpha_o\rho + \alpha_o(\alpha\beta)r] + \alpha_o(\alpha\beta)\tilde{\delta} + \alpha_o(\alpha\tilde{\epsilon})$$

$$= \rho + \alpha_o(\alpha\beta)\,[r + \tilde{\delta}] + \alpha_o(\alpha\tilde{\epsilon}) \ .$$

From Theorem 5, we know that $\|n\alpha^o - n\alpha^q\| \to 0(\frac{1}{n}) \to 0$ and $|\alpha_o^o - \alpha_o^q| \to 0(\frac{1}{n}) \to 0$ for any particular choice of m . If we substitute α^q for α^o in (8) , we can show that

$$n^2\gamma_i \approx n^2\beta_i(\alpha^q\beta)^2 E\{U'''(m + \tilde{y})\tilde{\delta}^2\} + n^2(\alpha_i^q)^2 E\{U'''(m + \tilde{y})\tilde{\epsilon}_i^3\} \ ,$$

to $0(\frac{1}{n})$. Since α^q is independent of m , Lemma 1 assures that $\|n^2\gamma\|$ is continuous in m and since the total error term is obtained by operating on γ by a linear functional uniformly bounded in n (see (10)), we need only show that the optimal choice of m for the particular utility function eventually lies in a compact domain.

Consider the auxiliary problem

$$\max_a \ E\{U(\rho + a[r + \tilde{\delta}])\} \ .$$

We will assume that this has a solution $\hat{a}$. Letting $\hat{m} \equiv \rho + \hat{a}r$, the principle of stochastic dominance asserts that $\tilde{R}$ is inferior to $\hat{m} + \hat{a}[r + \tilde{\delta}] = m + [\frac{\hat{m} - \rho}{r}]\tilde{\delta}$. But since diversification will allow the eventual convergence of $a\tilde{\epsilon}$, it is clear that $\tilde{R}^o \to \hat{m} + [\frac{\hat{m} - \rho}{r}]\tilde{\delta}$ (a.s.), and therefore, that $m \to \hat{m}$. Consequently, for any $\epsilon \geq 0$, there exists N such that $n > N$ implies that $|m - \hat{m}| \leq \epsilon$. <u>Q.E.D.</u>

It should also be clear from an examination of the proof of Theorem 6 that the existence of a risk-free asset played no crucial role. The pivotal assumption was that $E_i = \rho + r\beta_i$ for some ρ and r. In the absence of a risk-free asset, it can be shown that in the mean-variance world all efficient portfolios will be a linear combination of the same two portfolios. (This follows from the observation that the optimal portfolio of (1) is simply a linear combination of the two portfolios $V^{-1}E$ and $V^{-1}e$.) The portfolios may be chosen to be uncorrelated with each other and (approximately) we may choose one to be the minimum variance portfolio with $\alpha\beta = 0$ the "zero-beta" portfolio.[14] Letting the other portfolio be termed the "market portfolio" we can prove the following.

<u>Theorem 7</u>: If $E_i = \rho + r\beta_i$, then under the conditions of Lemma 1 other than (6) (b) the optimal portfolio will be a linear combination of two portfolios, one of which will approach the "zero-beta" portfolio and the other of which will approach the "market" portfolio to $0(1/n^2)$. Furthermore, as in Theorem 6, the proportion of wealth put in the "market" portfolio in the quadratic problem will converge to the exact proportion in the latter portfolio to $0(\frac{1}{n})$.

<u>Proof</u>: The proof is nearly identical to that of Theorem 6 and will be omitted. <u>Q.E.D.</u>

<u>Section III</u>

We now turn to an examination of capital market theory under the assumptions given above on the market model. The equilibrium theory of Lintner (1965) and Sharpe (1964) rests primarily on the twin assumptions that all investors in the market have identical subjective probability distributions (although this can be weakened (see Lintner [1971] and Ross [1971;3]) and that returns are evaluated solely in terms of two moments, their means and covariances (although the spread moment can easily be the stable Paretian α-parameter; see, for example Fama [1965, 1971]). The mean-variance assumption has been much attacked as being valid only under the very restrictive conditions that returns are normally distributed or that investors have quadratic utility functions. We will demonstrate that the theorems of Section II enable us to essentially drop the mean-variance assumption as necessary to the theory in large markets.

At this stage it is useful to recall some results from mean-variance capital market theory. Figure I depicts the familiar equilibrium situation in the presence of a risk-free asset yielding a return of ρ . The market portfolio with return E_m and risk σ_m is at the point of tangency with the market line. In equilibrium if all individuals have identical anticipations, they will divide their wealth between the riskless asset and this single risky market portfolio choosing some point on the market line. In equilibrium, then, the contribution of security i to the market portfolio must simply be $\alpha_i = p_i s_i / \sum_i p_i s_i$, i.e., the ratio of the value of security i , $p_i s_i$ to the total market value $\sum_i p_i s_i$, where $s_i \equiv$ number of shares of security i outstanding, and $p_i \equiv$ price per share of security i . In addition, it may be shown (see Sharpe [1970]) that in equilibrium

$$E_i - \rho = \frac{\sigma_{im}^2}{\sigma_m^2}(E_m - \rho) , \tag{11}$$

where $\sigma_{im}^2 \equiv$ covariance between security i and the market security, and $\sigma_m^2 \equiv$ variance of the market security.

If the (anticipated) returns on securities are given by the single-factor market model of Section II,

$$\tilde{X}_i = E_i + \beta_i\tilde{\delta} + \tilde{\epsilon}_i , \tag{5}$$

then (11) reduces to

$$E_i - \rho \approx \beta_i(E_m - \rho) , \tag{12}$$

where the approximation is of $0(\frac{1}{n})$.[15] In what follows we will prove that (5) implies (11) and (12).

First, though, we will use (9) to derive the basic equilibrium price relation (12) in an intuitive fashion that demonstrates the crucial role played by arbitrage. The arbitrage derivation lies at the heart of the robustness of the price relation (12). Consider an arbitrary portfolio formed by combining the risky assets:

$$\begin{aligned}\alpha\tilde{X} &= \alpha E + (\alpha\beta)\tilde{\delta} + \alpha\tilde{\epsilon} \\ &\approx \alpha E + (\alpha\beta)\tilde{\delta}\end{aligned}$$

where the portfolio, $\langle\alpha\rangle$, has been assumed to be sufficiently diversified that the independent element of risk, $\alpha\tilde{\epsilon}$, can be ignored in the sense of the previous sections.[16] Ignoring short sale restrictions, there is nothing to prevent us from choosing α so as to eliminate the systematic or factor risk as well, i.e., we may set

$$\alpha\beta = 0, \tag{13}$$

and the return on the portfolio

$$\alpha\tilde{X} \approx \alpha E,$$

a risk-free return. We could not form such a portfolio only if β were approximately a constant vector, but if β has sufficient variability we can simultaneously satisfy (13) and the definitional constraint for portfolios,

$$\alpha \mathbf{e} = 1,$$

without straining the requirement that α be diverse enough to allow $\alpha\tilde{\epsilon} \approx 0$.

However, if it is possible to attain a (nearly) risk-free return of αE and if there is a true risk-free asset yielding a return of ρ then simple arbitrage will require that in market equilibrium

$$\alpha E \approx \rho. \tag{14}$$

If $\alpha E \not\approx \rho$, then either the demand for risky securities if $\alpha E > \rho$ or the supply (short sale) if $\alpha E < \rho$ would be excessive for equilibrium.

Furthermore, (14) must hold for all portfolios sufficiently diverse to eliminate independent risk, $\alpha\tilde{\epsilon}$, along with factor or systematic risk, $\tilde{\delta}$. This will only occur if

$$E_i \approx \rho + a\beta_i,$$

where a is an arbitrary constant. By scaling $\tilde{\delta}$ appropriately we can normalize β so that $\alpha_m \beta = 1$ and

$$E_i \approx \rho + \beta_i \, (E_m - \rho). \tag{12}$$

One of the more remarkable features of (12), however, is that it will hold even if there is no true risk-free asset! Suppose that it is possible to form a risk-free portfolio with return $\approx \alpha E$. Another portfolio can be formed from the original one by adding the transactions vector, η, where

$$\eta \mathbf{e} = \eta\beta = 0, \tag{15}$$

and, therefore,

$$(\alpha+\eta)e = 1 \quad \text{and} \quad (\alpha+\eta)\beta = 0.$$

The return on this alternative portfolio will be given by $(\alpha+\eta)E = \alpha E+\eta E$. If $\alpha+\eta$ is also a well diversified portfolio so that $(\alpha+\eta)\tilde{\epsilon} \approx 0$, this return will also be (nearly) risk-free and once again an arbitrage situation arises if $\alpha E + \eta E \neq \alpha E$, or, if $\eta E \neq 0$. To insure that this holds for <u>all</u> alternative portfolios from (15) we must have

$$E_i \approx a + b\beta_i,$$

where a and b are constants. Summing over the market portfolio and normalizing $\alpha_m\beta = 1$, we can eliminate one constant and obtain

$$E_i \approx a + \beta_i(E_m - a).$$

The constant a can also be identified. If α is any "zero-beta" portfolio, i.e., $\alpha\beta=0$, then its return, $\alpha E = a$. Thus, all zero-beta portfolios have the same expected return.

In Section IV these findings are generalized to m-factor models, but for the moment we should stress that the arbitrage derivation of the market pricing relation aside from being the simplest available derivation has the additional virtue of implying the stability of the price relation in the strongest possible sense, as a necessary condition for the prevention of arbitrarily large (sure) returns. The above analysis, however, has largely been heuristic and to verify the results in a rigorous manner we can appeal to the analysis of the previous sections.

To begin with we define the notion of an ϵ-equilibrium. This concept was first introduced in game theory and has recently been studied in general equilibrium theory (see, e.g., Shapley and Shubik [1966] or Starr [1970]). We will adopt the following definition.

Definition :

An ϵ-equilibrium is defined as any set of prices $\langle P_1,\ldots,P_n\rangle$ such that $\|\langle D_1,D_2,\ldots,D_n\rangle\| \le \epsilon$ where $D_i \equiv$ the excess demand for asset i at prices $\langle P_1,\ldots,P_n\rangle$.

In essence, an ϵ-equilibrium is a set of asset prices (and, hence, expected returns) with the property that they almost clear the market. If the demand vector is continuous in the prices, then as $\epsilon \to 0$, any sequence of ϵ-equilibrium prices will converge to equilibrium prices (provided the latter exist and assuming prices are suitably normalized). We now prove the following formal theorem.

Theorem 8: If returns are subjectively viewed by investors as being generated by the market model, then the security line relation

$$E_i - \rho = \beta_i(E_m - \rho) \; , \tag{12}$$

where ρ is the return on the risk free asset, or equivalently, $E_i - \rho = \frac{\sigma^2_{im}}{\sigma^2_m}(E_m - \rho)$ will represent an ϵ-equilibrium with $\epsilon \to 0(\frac{1}{n})$. In addition, all investors will pick the same risky portfolio to $0(\frac{1}{n})$.

Proof: The proof proceeds by comparing the market excess demand that actually results when returns satisfy (12) with an artificial equilibrium that would occur if investors were forced to choose between the risk free asset and the quadratically optimal market portfolio. The first step is to demonstrate that the equilibrium values of risky assets V and V^q will be approximately equal in the two situations. Letting $\nu = \{1,\ldots,m\}$ index the investors in the economy, $\overline{W}_\nu$, the wealth (of agent ν) at the prices implicit in (12), W_ν, the dollar wealth placed at risk, and $\alpha_o(\nu)$, the proportion of wealth placed at risk, we have that

$$\begin{aligned} W_\nu &= \overline{W}_\nu \alpha_o(\nu) \\ &\equiv \overline{W}_\nu(\alpha_o^q(\nu) + \frac{\omega_\nu}{n}) \\ &= \overline{W}_\nu \alpha_o^q(\nu) + \overline{W}_\nu(\frac{\omega_\nu}{n}) \\ &= W_\nu^q + \overline{W}_\nu(\frac{\omega_\nu}{n}) \quad . \end{aligned}$$

Now,

$$V = \sum_\nu W_\nu = \sum_2 \overline{W}_\nu{}^{17}$$

and

$$V^q = \sum_\nu W_\nu^q \quad .$$

Hence,

$$\begin{aligned} |V - V^q| &= |\sum_\nu \overline{W}_\nu(\frac{\omega_\nu}{n})| \\ &= V|\sum_\nu \gamma_\nu \omega_\nu|\frac{1}{n} \\ &\equiv V\frac{\omega}{n} \quad , \end{aligned}$$

where $\gamma_\nu \equiv \dfrac{\overline{W}_\nu}{V}$, and $\omega \equiv |\sum_\nu \gamma_\nu \omega_\nu|$ is, therefore, a convex combination of the ω_ν difference terms. At the extreme, then, ω will be bounded by $\sup_\nu |\omega_\nu|$ for any n and since, from Theorem 6, $\sup_n |\omega_\nu|$ is bounded for each ν, $\omega \le \sup_\nu \sup_n |\omega_\nu| < +\infty$. Of course, in practice, with diverse investors many of the terms of ω will cancel and to the extent that wealth is not concentrated, we would expect ω to be considerably less than its bound, $\sup_\nu \sup_n |\omega_\nu|$.

Letting $p_i s_i$ represent the dollar value of the outstanding stock of security i at the prices implicit in (12), we know from capital market theory that at these prices the quadratic system is in equilibrium with demand equal to supply, i.e., $V^q \alpha_i^q = p_i s_i$. Hence, the normed difference between the actual demand at these prices and the supply is given by

$$\begin{aligned} \|D - S\| &= \| \sum_\nu W_\nu \alpha(\nu) - V^q \alpha^q \| \\ &= \|V \sum_\nu \gamma_\nu \alpha_o(\nu) \alpha(\nu) - V^q \alpha^q \| \\ &= \|V \sum_\nu \gamma_\nu \alpha_o(\nu)(\alpha(\nu) - \alpha^q) + (V - V^q)\alpha^q \| \\ &\le V \| \sum_\nu \gamma_\nu \alpha_o(\nu)(\alpha(\nu) - \alpha^q)\| + |V - V^q| \, \|\alpha^q\| , \end{aligned}$$

where we have made use of the triangle inequality. From Theorem 6 we have $\|\alpha(\nu) - \alpha^q\| = a_\nu / n^\nu$, where a_ν is bounded in n, and $\|\alpha^q\| = \frac{b}{n}$ where b is also bounded in n. It follows that

$$\begin{aligned} \|D - S\| &\le V \| \sum_\nu \gamma_\nu \alpha_o(\nu) \frac{a_\nu}{n^2} \| + V \frac{\omega}{n} \frac{b}{n} \\ &\equiv \frac{V}{n} \frac{(\hat{a} + \omega b)}{n} , \end{aligned}$$

where $\hat{a} \equiv |\sum_{\nu} \gamma_{\nu} \alpha_o(\nu) a_{\nu}|$ is bounded by a similar argument as that which showed ω to be bounded. Since V/n is bounded (note that V^q/n is bounded, assuming the series of stocks of assets is bounded), $\|D - S\|$ is of order $\frac{1}{n}$. The assertion that all investors will pick identical portfolio to $0(\frac{1}{n})$ now follows directly from Theorem 6 and the capital market separation theorem in the quadratic case.

Q.E.D.

In the absence of a risk-free security the result obtained in Theorem 8 will still hold, except ρ must now be reinterpreted in the mean-variance capital market theory to be the rate of return on a portfolio uncorrelated with the market portfolio.[18] Theorem 9 asserts that equation (12) will hold in equilibrium, in an approximate sense, not only in the absence of a risk-free asset but also in general security markets given only identical subjective probability distributions and a market model.

Theorem 9: If returns are subjectively viewed by investors as being generated by the market model, then even in the absence of a risk-free asset the security line relation

$$E_i - \rho = \beta_i (E_m - \rho) \qquad (12)$$

will represent an ϵ-equilibrium with $\epsilon \to 0(\frac{1}{n})$. In addition, all investors will form their portfolios as linear combinations of the same two portfolios, a market portfolio and a "zero-beta" portfolio, to $0(\frac{1}{n})$.

Proof: Nearly identical to that of Theorem 8 with Theorem 7 used in place of Theorem 6. The two portfolios are identified by an examination of (1); see Black [1971] for details.

Q.E.D.

The individual portfolio theorems contain the kernel behind the general validity of the security line equilibrium equation (12) in large markets. Essentially, if the security line relation did not approximately hold, then individuals would be able to diversify away all risk and attain arbitrarily large returns as the number of securities increased. This would be incompatible with equilibrium.

IV. Generalizations

The theorems of the above sections were derived on the assumption that returns were (ex ante) subjectively viewed as being generated by single-factor market models. In general, though, it can be shown that the above results remain valid whenever the number of factors is small relative to the number of securities. The principal exceptions can best be illustrated with the two-factor model. Suppose, first, that returns are generated by a market model of the form

$$\tilde{X}_j = E_j + \beta_{j1}\tilde{\delta}_1 + \beta_{j2}\tilde{\delta}_2 + \tilde{\epsilon}_j \ .$$

If the relevant series $\langle E_j \rangle$, $\langle \beta_{j1} \rangle$, $\langle \beta_{j2} \rangle$, and $\langle \sigma_j \rangle$ are sufficiently variable in a sense analogous to that of Lemma 1 yet appropriately bounded, then it will still be the case that as n gets large the AQP solution α^q will approach the solution to the exact problem α^o . Furthermore, the level of risk required to attain any given return m will approach zero as the distribution of $(\alpha\beta_1)\tilde{\delta}_1 + (\alpha\beta_2)\tilde{\delta}_2 + \alpha\tilde{\epsilon} \to 0$, (a.s.) , and as $\alpha E \to m$.

However, it will no longer be true that $\alpha^o \to \alpha^q$ when the mean series is not sufficiently variable and this was the case which was relevant for the study of capital market equilibrium. In quadratic equilibrium in the capital market (with or without a riskless asset) the security line relation

$$E_j - \rho = \frac{\sigma_{jm}^2}{\sigma_m^2} (E_m - \sigma) \tag{16}$$

$$\approx (\gamma_1 \beta_{j1} + \gamma_2 \beta_{j2}) (E_m - \rho) , \tag{17}$$

to $0(\frac{1}{n})$ where $\gamma_i \equiv \sigma_i^2/(\sigma_1^2 + \sigma_2^2)$ and $\sigma_m^2 \approx \sigma_1^2 + \sigma_2^2$ with $(\alpha\beta_1)$ and $(\alpha\beta_2)$ normalized to unity.(19) Thus, the only difference between the single and multiple factor cases is that the single volatility index is replaced by a convex combination of the individual volatility indexes with weights proportional to the variance of the index. Now consider the individual portfolio problem if (16) is satisfied. The investor will obtain a return of

$$\begin{aligned} \tilde{R} &= \alpha_o \rho + \alpha E + (\alpha\beta_1)\tilde{\delta}_1 + (\alpha\beta_2)\tilde{\delta}_2 + \alpha\tilde{\epsilon} \\ &= \rho + [(\alpha\beta_1)\gamma_1 + (\alpha\beta_2)\gamma_2] (E_m - \rho) \\ &\quad + (\alpha\beta_1)\tilde{\delta}_1 + (\alpha\beta_2)\tilde{\delta}_2 + \alpha\tilde{\epsilon} \\ &\rightarrow \rho + [(\alpha\beta_1)\gamma_1 + (\alpha\beta_2)\gamma_2] (E_m - \rho) \\ &\quad + (\alpha\beta_1)\tilde{\delta}_1 + (\alpha\beta_2)\tilde{\delta}_2 , \text{ (a.s.)} , \end{aligned}$$

for well diversified portfolios. As in the single-factor model the investor is unable to eliminate systematic risk in equilibrium. In the single-factor model, though, the mean return desired by the investor would also specify the level of systematic risk borne and the choice of the optimal risky portfolio would simply be governed (for large n) by the desire to eliminate the unsystematic risk component. In the two (or many) factor model the specification of a mean return level (as the outcome of the exact expected utility maximization) constrains $\gamma_1(\alpha\beta_1) + \gamma_2(\alpha\beta_2)$, but the final choice of $(\alpha\beta_1)$ and $(\alpha\beta_2)$ will depend upon the investor's risk preferences and will not simply be guided by the relation between σ_1 and σ_2 as in the minimum variance problem. As a consequence, even with large numbers of securities and even in the presence of a risk-free asset, investors will not in general purchase the same portfolio of risky assets. Each investor will now act, for large n , approximately to minimize the nonsystematic risk $\alpha V \alpha$, subject to two constraints $\alpha\beta_1 = m_1$ and $\alpha\beta_2 = m_2$, where m_1 and m_2 emerge from his risk attitudes concerning $\tilde{\delta}_1$ and $\tilde{\delta}_2$. The solution to the problem will take the form

$$\alpha = \lambda_1(V^{-1}\beta_1) + \lambda_2(V^{-1}\beta_2) + \theta(V^{-1}e). \tag{18}$$

where λ_1 , λ_2 , and θ are Lagrange multipliers. Equation (18) can be interpreted to mean that optimal portfolios will be formed by individuals as linear combinations of two portfolios taken with borrowing or lending at the riskless rate ρ (or the return on the minimum variance, "zero-beta" $\alpha\beta_1 = \alpha\beta_2 = 0$ portfolio in the absence of a riskless asset). By a straight-forward extension of the arbitrage argument of Section III or, more formally, of Theorems 6, 7, 8, and 9, it can again be shown that the security line (17) will represent an ϵ-equilibrium in the capital market of $O(\frac{1}{n})$, although the equilibrium γ_i weights need not correspond to $(\sigma_i^2/\sigma_1^2 + \sigma_2^2)$.

For the general case, when the number of factors ℓ is considerably less than the number of securities n, the security line equation

$$E_i - \rho = (\gamma_1 \beta_{1i} + \dots + \gamma_\ell \beta_{\ell i})(E_m - \rho) ,$$

where $\gamma_1 + \dots + \gamma_\ell = 1 (\gamma_j \geq 0)$ will obtain. The return E_m is again the return on a market portfolio with weights proportional to the value shares of the individual securities, but the market portfolio no longer enjoys the separation property, and the capital market line for efficient portfolios no longer exists.[20]

V. Summary and Conclusions

The above sections have shown that neither the assumption that returns are normally distributed nor the assumption that utility functions are quadratic is critical to the validity of either the separation theorem or the mean-variance capital market equilibrium model. In a single-factor model with a large number of securities, individual portfolio behavior was unaltered, and the equilibrium theory remained intact for general factor models. In particular, as long as the degree of dependence between anticipated asset returns is not too high, the security line equation of the capital market model will become an increasingly good approximation to equilibrium as the number of securities becomes large (relative to the number of factors in the market model). In a world with only 1,000 securities (and a single-factor market model), the security line equation would depart from an equilibrium at most to the order of 1/1000 of the outstanding value of each security, and, in general, the approximation would be considerably better. This fundamental price equation emerges as a consequence of simple portfolio arbitrage possibilities and it is this viewpoint that accounts for its robustness to alternative specifications of the underlying model.

Appendix

Theorem 1: If $\langle\sigma_j\rangle$ and $\langle E_j\rangle$ are uniformly bounded, $\langle B_n\rangle$ is uniformly bounded away from zero, and

$$E\{\tilde{y}_n \mid y_1, \ldots, y_{n-1}\} = 0 \; , \qquad A(1)$$

then

$$\alpha^q \tilde{y} \to 0 \; , \; (a.s.) \quad .$$

Proof:

Condition (1) is simply a generalization of the notion of independence and is equivalent to the assertion that the partial sums $S_n \equiv y_1 + \ldots + y_n$, form a strict martingale. It is also easy to show that the covariance matrix V_n is diagonal under (1). The need for the boundedness conditions will become apparent in the course of the proof.

Let the bounds be given by

$$B_n \geq b > 0 \; ,$$

$$e^{-1} \geq \sigma_j^2 \geq f^{-1} > 0 \; , \qquad A(2)$$

and

$$|E_j| \leq a < +\infty \; .$$

From (3) and (4) we have

$$\Delta = - A_n^2 B_n \; ,$$

and upon applying some algebra we obtain

$$\alpha_i^q = \sigma_i^{-2}(\lambda E_i + \theta) = (\sigma_i^{-2}/A_n B_n)\varphi_i \ , \tag{A(3)}$$

where

$$\varphi_i \equiv E_i\left(\frac{1}{A_n} \sum_j E_j\sigma_j^{-2} - m\right) + m\frac{1}{A_n}\sum_j E_j\sigma_j^{-2} - \frac{1}{A_n}\sum_j E_j^2\sigma_j^{-2}$$

$$= E_i(\bar{E} - m) + m\bar{E} - (B_n + \bar{E}^2)$$

$$= (E_i - \bar{E})(\bar{E} - m) - B_n \ ;$$

$$\bar{E} \equiv \frac{1}{A_n}\sum_{j=1}^{n} E_j\sigma_j^{-2} \ .$$

By the bounds on $\langle E_j\rangle$ and $\langle B_n\rangle$ we have

$$-C - b \le \varphi_i \le C \ , \tag{A(4)}$$

where

$$C \equiv \mathrm{Max}\left\{\left|\tfrac{1}{4}(a - m)^2\right| , \left|\tfrac{1}{4}(m - 3a)(a - 3m)\right|\right\}^{(1)}$$

From the assumption that $\langle\sigma_j\rangle$ is uniformly bounded we have

$$\left(\frac{e}{f}\right)\frac{1}{n} \le \frac{\sigma_i^{-2}}{A_n} \le \left(\frac{f}{e}\right)\frac{1}{n} \ ,$$

and combining this with A(2), A(3), and A(4) yields

$$
\begin{aligned}
|\alpha_i^q| &= |(\sigma_i^{-2}/A_n B_n)\varphi_i| \\
&\leq [(c+a)\frac{f}{eb}]\frac{1}{n} \\
&\equiv \frac{k}{n} .
\end{aligned}
$$

We can now apply a well-known statement of the strong law of large numbers (see Loeve, p. 387). Let $\tilde{Z}_{ni} \equiv n\alpha_i^q\tilde{y}_i$. Since

$$
\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^{n} E\{z_{ni}^2\} &= \sum_{i=1}^{n} (\alpha_i^q)^2\sigma_i^2 \\
&\leq \frac{1}{e}\sum_{i=1}^{n} (\alpha_i^q)^2 \\
&\leq \frac{k^2}{e}\cdot\frac{1}{n} \to 0 ,
\end{aligned}
$$

it follows from the limit theorem that

$$
\begin{aligned}
S_{nn} &\equiv \sum_{i=1}^{n} \alpha_i^q\tilde{y}_i \\
&= \frac{1}{n}\sum_{i=1}^{n} \tilde{Z}_{ni} \\
&\to 0 , \text{ (a.s.)} .
\end{aligned}
$$

<u>Q.E.D.</u>

Lemma 1:

If we assume that $\langle B_j \rangle$, $\langle E_j \rangle$, and $\langle \sigma_j \rangle$ are bounded as in Theorem 1, that $e'V^{-1}E$, $e'V^{-1}\beta$, $\beta'V^{-1}E$, $\beta'V^{-1}\beta$, and $e'V^{-1}e$ are of the same order and that there exist positive constants a and b such that

$$(e'V^{-1}e)^{-1}\{\beta'V^{-1}\beta)\ (e'V^{-1}e) - (e'V^{-1}\beta)^2\} \geq a(e'V^{-1}e) \ , \qquad (6)(a)$$

and

$$B \equiv (e'M^{-1}e)^{-2}\{(E'M^{-1}E)\ (e'M^{-1}e) - (e'M^{-1}E)^2\} \geq b \ , \qquad (6)(b)$$

then

$$\alpha^q \rightarrow 0(\frac{1}{n}) \rightarrow 0$$

and

$$\alpha^q\beta \rightarrow 0(\frac{1}{n}) \rightarrow 0 \ .$$

Proof:

The proof is similar to that of Theorem 1, and we will do in detail only those portions that are distinct. Since $\langle \sigma_j \rangle$ is bounded, $(e'V^{-1}e) \rightarrow 0(\frac{1}{n})$ and we first must show that $(e'M^{-1}e)^{-1} \rightarrow 0(\frac{1}{n})$. Observe that

$$M^{-1} = V^{-1} - \pi V^{-1}\beta\beta'V^{-1} \ ,$$

where

$$\pi \equiv \sigma^2/[1 + \sigma^2(\beta'V^{-1}\beta)] \ .$$

---

(1) The bounds are obtained by considering the extreme values of the form $(E_i - \overline{E})(\overline{E} - m)$ and allowing both E_i and $\overline{E}$ to take on any values between $-a_2$ and $+a$. Tighter bounds could be found by explicitly considering the $\langle \sigma_j^2 \rangle$ series, but the effort would not warrant the reward.

Hence, from (6) (a),

$$e'M^{-1}e = e'[V^{-1} - (\sigma^2/(1 + \sigma^2\beta'V^{-1}\beta))V^{-1}\beta\beta/V^{-1}]\ e$$

$$= (1 + \sigma^2\beta'V^{-1}\beta)^{-1}\{e'V^{-1}e) + \sigma^2[(\beta'V^{-1}\beta)(e'V^{-1}e)$$

$$- (e'V^{-1}\beta)^2]\}$$

$$\geq \left[\frac{1 + \sigma^2 a(e'V^{-1}e)}{1 + \sigma^2(\beta'V^{-1}\beta)}\right] (e'V^{-1}e)$$

$$\geq \left[\frac{1 + \sigma^2 a(e'V^{-1}E)}{1 + \sigma^2 c(e'V^{-1}e)}\right] (e'V^{-1}e)$$

$$\geq \left[\frac{\sigma^2 a}{1 + \sigma^2 c}\right] (e'V^{-1}e) ,$$

where $c \equiv \sup_n [(\beta'V^{-1}\beta)/(e'V^{-1}e)] < +\infty$ by assumption.

From (2), as in Theorem 1, we have

$$-\Delta = A^2B ,$$

where now

$$A \equiv (e'M^{-1}e) ,$$

and using this result we can derive the formula analogous to A(3),

$$\alpha_i = \frac{\sigma_i^{-2}}{A} \quad \frac{1}{B}\varphi_i , \qquad \text{A(5)}$$

where

$$\varphi_i = (a_i - b_i \bar{E})(\bar{E} - m) - b_i B ,$$

$$\bar{E} \equiv (e'M^{-1}e)^{-1} e'M^{-1}E ,$$

$$a_i \equiv E_i - \beta_i \left[\frac{\beta' V^{-1} E}{\sigma^{-2} + \beta' V^{-1} \beta} \right] ,$$

and

$$b_i \equiv 1 - \beta_i \left[\frac{\beta' V^{-1} e}{\sigma^{-2} + \beta' V^{-1} \beta} \right] .$$

Now, since $\langle E_i \rangle$ and $\langle \beta_i \rangle$ are bounded and since $(\beta' V^{-1} E)$ and $(\beta' V^{-1} e)$ are assumed to be of no greater order than $\beta' V^{-1} \beta$, $\langle a_i \rangle$, and $\langle b_i \rangle$ will also be (uniformly) bounded. Similarly,

$$e'M^{-1}E = e'V^{-1}E - \pi(e'V^{-1}\beta)(\beta'V^{-1}E)$$

$$= e'V^{-1}E - \frac{(e'V^{-1}\beta)(\beta'V^{-1}E)}{(\sigma^{-2} + \beta'V^{-1}\beta)} ,$$

and the bounds on $\langle E_j \rangle$ assure that this term is of no greater order than $\beta' V^{-1} \beta$ and that $\bar{E}$ is, therefore, also bounded. It follows that φ_i / B is bounded and that the order of α_i^q is determined solely by $[\sigma_i^{-2}/A]$. But since we have shown that $e'M^{-1}e \geq [\sigma^2 a/1 + \sigma^2 c](e'V^{-1}e)$ and since $\langle \sigma_j \rangle$ is bounded above and away from zero, $\alpha_i^q \to 0(\frac{1}{n}) \to 0$.

The proof that the component of systematic risk $\alpha' \beta \to 0(\frac{1}{n}) \to 0$ is straightforward. From A(5)

$$\alpha'\beta = \frac{1}{AB}\beta'V^{-1}\{(\bar{E} - m)(E - \beta\left[\frac{\beta'V^{-1}E}{\sigma^{-2} + \beta'V^{-1}\beta}\right]$$

$$- \bar{E}e + \bar{E}\beta\left[\frac{\beta'V^{-1}E}{\sigma^{-2} + \beta'V^{-1}\beta}\right])$$

$$- Be + B\beta\left[\frac{\beta'V^{-1}e}{\sigma^{-2} + \beta'V^{-1}\beta}\right]\}$$

$$= \frac{1}{AB}(\bar{E} - m)\ (\beta'V^{-1}E - \beta'V^{-1}e)\ (1 - \left[\frac{\beta'V^{-1}\beta}{\sigma^{-2} + \beta'V^{-1}\beta}\right])$$

$$- \frac{1}{A}(\beta'V^{-1}e)\ (1 - \left[\frac{\beta'V^{-1}\beta}{\sigma^{-2} + \beta'V^{-1}\beta}\right]) \ .$$

Since $A \equiv e'm^{-1}e \geq [\sigma^2 a/1 + \sigma^2 c]\ (e'V^{-1}e)$ and since $B \geq b$ and $\bar{E}$ is bounded, the first term is of the order of $(1 - [\beta'V^{-1}\beta/\sigma^{-2} + \beta'V^{-1}\beta])$ and similarly, the boundedness of $(\beta'V^{-1}e)/(e'V^{-1}e)$ assures that the second term, too, is of the same order. However,

$$1 - \left[\frac{\beta'V^{-1}\beta}{\sigma^{-2} + \beta'V^{-1}\beta}\right] = \frac{\sigma^{-2}}{\sigma^{-2} + \beta'V^{-1}\beta} \to 0(\tfrac{1}{n}) \to 0 \ .$$

<u>Q.E.D.</u>

<u>Lemma 2</u>: The optimal solution α^o and $\alpha^o\beta$ are bounded of order $1/n^{\frac{1}{2}}$, i.e.,

$$||\, n^{\frac{1}{2}}\alpha^o ||$$

and

$$|n^{\frac{1}{2}}\alpha^o\beta| \quad \text{are bounded.}$$

<u>Proof</u>: By Taylor's theorem,

$$E\{U(\alpha\tilde{x})\} = U(m) + E\{U'(m)((\alpha\beta)\tilde{\delta} + \alpha\tilde{\epsilon})\} + \tfrac{1}{2}E\{U''(m + \tilde{y})((\alpha\beta)\tilde{\delta} + \alpha\tilde{\epsilon})^2\} ,$$

or

$$U(m) - E\{U(\alpha\tilde{x})\} = -\tfrac{1}{2}E\{U''(m + \tilde{y})((\alpha\beta)\tilde{\delta} + \tilde{\epsilon}))^2\} ,$$

where

$$\tilde{y} \in [0, (\alpha\beta)\tilde{\delta} + \alpha\tilde{\epsilon}] .$$

From Lemma 1, $\alpha^q \to 0(\frac{1}{n})$ and $(\alpha^q\beta) \to 0(\frac{1}{n})$, and Lemma 2A in the appendix thus implies that

$$U(m) - E\{U(\alpha^q\tilde{x})\} \to 0(\frac{1}{n}) .$$

Assume now, contrary to the porposition that $\|n^{\frac{1}{2}}\alpha^o\|$ or $|n^{\frac{1}{2}}\alpha^o\beta|$ are unbounded. By the same reasoning as in the appendix

$$-\tfrac{1}{2}n\, E\{U''(m + \tilde{y})((\alpha^o\beta)\tilde{\delta} + \alpha^o\tilde{\epsilon})^2\}$$

will also be unbounded and

$$n[U(m) - E\{U(\alpha^o\tilde{x})\}]$$

will diverge.[2] It follows that there will exist some N such that for $n > N$

$$E\{U(\alpha^q\tilde{x})\} > E\{U(\alpha^o\tilde{x})\} ,$$

contradicting the optimality of α^o .

Q.E.D.

---

[2] The reader need only observe that since $\beta\beta'$ is positive semi-definite and since the lemma implies that orthogonal cross-terms are negligible,

$$\begin{aligned}
&-\tfrac{1}{2}E\{N''(m + \tilde{y})((\alpha^o\beta)\tilde{\delta} + \tilde{\epsilon})^2\}n \\
&\to (\alpha^o[-\tfrac{1}{2}E\{N''(m + \tilde{y})\tilde{\delta}^2\}\beta\beta' \\
&\quad -\tfrac{1}{2}E\{N''(m + \tilde{y})\tilde{\epsilon}\tilde{\epsilon}'\}]\alpha^o)n \\
&\geq (-\tfrac{1}{2}E\{N''(m + \tilde{y})(\alpha^o\tilde{\epsilon})^2\})n ,
\end{aligned}$$

Lemma 1A: If ϵ is a mean zero random variable and $H(\cdot)$ is a twice differentiable function, then

$$E\{H(\frac{\epsilon}{n})\epsilon\} \to 0(\frac{1}{n})$$

for large n .

Proof: By Taylor's theorem

$$H(\frac{\epsilon}{n}) = H(0) + H'(0)\frac{\epsilon}{n} + \frac{1}{2}H''(y)\frac{\epsilon^2}{n^2} ,$$

where $y \in [0, \frac{\epsilon}{n}]$. Hence,

$$E\{H(\frac{\epsilon}{n})\epsilon\} = E\{[H(0) + H'(0)\frac{\epsilon}{n} + \frac{1}{2}H''(y)\frac{\epsilon^2}{n^2}]\epsilon\}$$

$$= H'(0)\frac{\sigma^2}{n} + \frac{1}{n^2}E\{\frac{1}{2}H''(y)\epsilon^3\} .$$

Let

$$\bar{H}(\frac{\epsilon}{n}) \equiv \sup_{y \in [0, \frac{\epsilon}{n}]} |\frac{1}{2}H''(y)| .$$

Clearly, $(\forall \epsilon)$ $\bar{H}(\frac{\epsilon}{n})$ is a declining function of n ; hence, for $n > 1$

$$\bar{H}(\frac{\epsilon}{n}) \le \bar{H}(\epsilon)$$

and

---

which diverges if $\|n^{\frac{1}{2}}\alpha^o\|$ is unbounded. Similarly, $(\alpha^o\beta)^2 n$ must also be bounded to prevent divergence.

$$
\begin{aligned}
&|E\{\tfrac{1}{2} H''(y)\epsilon^3\}| \\
&\quad = \left| \int_{-\infty}^{\infty} \frac{1}{2} H''(y)\, \epsilon^3 dF_\epsilon \right| \\
&\quad \leq - \int_{-\infty}^{0} \overline{H}(\epsilon)\, \epsilon^3 dF_\epsilon + \int_{0}^{\infty} \overline{H}(\epsilon)\, \epsilon^3 dF_\epsilon \\
&\quad \equiv c .
\end{aligned}
$$

Note that c is independent of n, and we will assume that $H(\cdot)$ is chosen so as to make $c < \infty$. It follows that

$$
\begin{aligned}
\left| E\{H(\tfrac{\epsilon}{n})\, \epsilon\} - H'(0) \frac{\sigma^2}{n} \right| &= \frac{1}{n^2} \left| E\{\tfrac{1}{2} H''(y)\, \epsilon^3\} \right| \\
&\leq \frac{c}{n^2} ,
\end{aligned}
$$

which proves the theorem, Q. E. D.

The proofs of the theorems in the text rely on an extension of Lemma 1A that we will now prove.

Lemma 2A: Let $\langle \epsilon_i, \epsilon_j, \epsilon_k \rangle$ be mean zero random variables that are independent if their subscripts differ. If $\langle \alpha_i, \alpha_j, \alpha_k \rangle$ are contants and $H(\cdot)$ is sufficiently continuously differentiable, then

$$
E\{H(\frac{\alpha_i}{n}\epsilon_i + \frac{\alpha_j}{n}\epsilon_j)\, \epsilon_i \epsilon_j\} \to \begin{cases} \text{constant if } i=j \\ 0(\frac{1}{n^2}) \text{ if } i \neq j \end{cases}
$$

$$
E\{H(\frac{\alpha_i}{n}\epsilon_i + \frac{\alpha_j}{n}\epsilon_j + \frac{\alpha_k}{n}\epsilon_k)\, \epsilon_i \epsilon_j \epsilon_k\} \to \begin{cases} \text{constant if } i=j=k \\ 0(\frac{1}{n}) \text{ if only two are identical} \\ 0(\frac{1}{n^3}) \text{ if all three differ.} \end{cases}
$$

<u>Proof</u>: We will treat the three cases in succession. If i=j=k, then

$$E\{H(\frac{\alpha}{n}\epsilon)\epsilon^3\} \to H(0)E\{\epsilon^3\} ,$$

which is a constant independent of n .[3] If two differ, then we must consider the Taylor expansion

$$E\{H(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2)\epsilon_1\epsilon_2^2\} = E\{H(0)\epsilon_1\epsilon_2^2\} + E\{H'(0)[\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2]\epsilon_1\epsilon_2^2\}$$

$$+ E\{\frac{1}{2}H''(y)[\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2]^2\epsilon_1\epsilon_2^2\}$$

$$= \frac{\alpha_1}{n}H'(0)\, E\{\epsilon_1^2\epsilon_2^2\}$$

$$+ \frac{1}{n^2}E\{\frac{1}{2}H''(y)[\alpha_1^2\epsilon_1^3\epsilon_2^2$$

$$+ 2\alpha_1\alpha_2\epsilon_1^2\epsilon_2^3 + \alpha_2^2\epsilon_1\epsilon_2^4]\} ,$$

where $y \in [0, \frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2]$. As in the proof of Lemma 1A we can bound the final term and obtain

$$E\{H(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2)\epsilon_1\epsilon_2^2\} \to O(\frac{1}{n}) .$$

Finally, if all three differ, we have

---

[3] If $E\{\epsilon^3\} = 0$, then Lemma 1A would imply that $E\{H(\frac{\alpha}{n})\epsilon)\epsilon^3\} \to O(\frac{1}{n}) \to 0$.

$$E\{H(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2 + \frac{\alpha_3}{n}\epsilon_3)\epsilon_1\epsilon_2\epsilon_3\}$$

$$= E\{H(0)\epsilon_1\epsilon_2\epsilon_3\} + E\{H'(0)(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2 + \frac{\alpha_3}{n}\epsilon_3)\epsilon_1\epsilon_2\epsilon_3\}$$

$$+ E\{\frac{1}{2}H''(0)(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2 + \frac{\alpha_3}{n}\epsilon_3)^2\epsilon_1\epsilon_2\epsilon_3\}$$

$$+ E\{\frac{1}{6}H'''(0)(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2 + \frac{\alpha_3}{n}\epsilon_3)^3\epsilon_1\epsilon_2\epsilon_3\}$$

$$+ E\{\frac{1}{24}H''''(y)(\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2 + \frac{\alpha_3}{n}\epsilon_3)^4\epsilon_1\epsilon_2\epsilon_3\}$$

$$= \frac{a}{n^3} + \frac{b}{n^4}$$

$$\rightarrow 0(\frac{1}{n^3})$$

where $y \in [0, (\frac{\alpha_1}{n}\epsilon_1 + \frac{\alpha_2}{n}\epsilon_2 + \frac{\alpha_3}{n}\epsilon_3)]$ and a and b are bounded by arguments identical to those given above. The proof of the two-variable case is now obvious and will be omitted.

<u>Q.E.D.</u>

Notice that the proof of the above lemma and, hence, of the theorems in the text requires only that the random variables be mean independent, i.e., $E\{\epsilon_i \mid < \epsilon_1, \ldots, \epsilon_{i-1} > \} = 0$, but for ease of exposition we have sacrificed the slight increase in generality. In addition, note that Lemma 2A generalizes in a straightforward fashion to the case where $\langle \alpha_i, \alpha_j, \alpha_k \rangle$ lies in a bounded domain or where the base order is $1/n^\beta$ rather than $1/n$. Finally, it should be clear that Lemma 2A generalizes in an obvious fashion to the case of m random variables $\langle \epsilon_1, \ldots, \epsilon_m \rangle$.

Footnotes

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[1] Samuelson [1970] has verified an approximation of this sort for the case where the range of the random variables converges to zero in an appropriate limiting process.

[2] The constraint $\alpha \geq 0$ is also often imposed to prevent short sales, but we will not worry about this restriction. Its imposition would complicate the derivation of our theorems and in some cases would weaken the error bounds obtained but would not substantially alter the main conclusions.

Also, throughout we will assume that the utility function is sufficiently regular to permit the proofs to go through (this will generally mean that it must be continuously differentiable of some finite order).

[3] As n grows the dimension of α^o and α^q increases, and the mathematics requires that we view them as elements of an infinite dimensional (sequence) space. Any of the ℓ_p norms, for example, would now do but the ℓ_∞ uniform metric, although the weakest, is also one of the most intuitive. We will occasionally indicate below where the choice of a norm will make an important difference.

[4] In what follows we will always assume that the random variables possess second moments. This restriction can be removed and the results easily extended to the case where absolute moments above the first exist; however, the AQP solution will be the solution to an α-moment problem (see Fama [1971]) as in the stable Paretian theory.

[5] If V_n is singular, V_n^{-1} will denote the generalized inverse of V_n. The α^q portfolio is now indeterminate, and many choices will be indifferent solutions.

[6] This is the same as almost everywhere (a.e.) convergence (see Loeve [1963]) and implies weak convergence. In particular, it implies that for every concave $U(\cdot)$, $E\{U(\alpha^q \tilde{y})\} \to U(0)$.

[7] We will use the order notation $X_n \to 0(\frac{1}{n^\alpha})$ to mean that $\|n^\alpha X_n\|$ is bounded. Notice that this implies that for any $\beta < \alpha$, $\|n^\beta X_n\| \to 0$ and that $X_n \to 0(\frac{1}{n^\beta})$. Some simple algebra is also useful:

(i) if $X_n \to 0(\frac{1}{n^\alpha})$ and $Y_n \to 0(\frac{1}{n^\beta})$, then $X_n Y_n \to 0(\frac{1}{n^{\alpha+\beta}})$.

(where X_n and Y_n are scalars), and

(ii) if $X_n \to 0(\frac{1}{n^\alpha})$ and $Y_n \to 0(\frac{1}{n^\beta})$ and $\alpha > \beta$, then

$$X_n + Y_n \to 0(\frac{1}{n^\beta}) .$$

Finally, we will say two sequences $\langle X_n \rangle$ and $\langle Y_n \rangle$ are of the same order if the sequence $\langle | X_n / Y_n | \rangle$ is bounded above and away from zero.

[8] This condition can be weakened. For example, the result is unaffected if for some subsequence of the securities $\sigma_i^{-2}/A_n \to 0(1/n)$. All wealth would then be concentrated in this subsequence.

[9] Getting ahead of the story a bit, since the nature of $\tilde{y}_1$ and $\tilde{y}_2$ is arbitrary, it is clear that this term will not, in general, stochastically dominate all other portfolios and there will exist utility functions which will yield different optimal portfolios.

[10] There is no loss of generality in specifying $\tilde{\delta}$ to have zero mean since its mean return is already incorporated in E .

[11] The slash over a vector indicates its transpose, and we will continue to use this notation only when the form is ambiguous.

[12] For large n , the E vector will act like a constant vector plus $r\beta$ and $B \to 0$ as $(e'V^{-1}e)^{-1}$.

[13] This is one of those cases where the norm used is critical. For example, the approximation $n\alpha^o \approx n\alpha^q$ is not generally valid in the ℓ_1 norm $\|x\| \equiv \sum_i |x_i|$.

[14] This point has also been made by Fischer Black [1971] and the reader is referred to Black's paper for a detailed examination of the mean-variance case.

[15] Simply note that $\sigma_{im}^2 \equiv \mathrm{Cov}\,(\alpha\tilde{X},\tilde{X}_i) = (\alpha\beta)\beta_i\sigma_\delta^2 + \alpha_i\sigma_i^2 \approx \beta_i\sigma_m^2$ where without loss of generality we may normalize $\tilde{\delta}$ so as to set $\alpha\beta = 1$.

[16] If there is no independent element of risk, i.e., $\tilde{\epsilon}_i \equiv 0$, then the analysis which follows is exact and independent of n (≥ 2). The model may now be considered to be isomorphic to a degenerate state-space model, however, the argument is more general than that of Beja [1971] , say, since we have no need for markets in contingent claims. This isomorphism between factor and state-space models is of interest in its own right and will be the subject of another paper.

[17] This assumes the existence of no capital assets yielding the risk free return ρ and that all assets are included in the universe of securities under consideration. The proof can easily be modified to handle the alternative situation.

[18] See Fischer Black [1971].

[19] In a number of interesting cases this particular normalization will not be possible. If, for example, one of the portfolios has the "zero-beta" property, $\alpha\beta_i = 0$, we cannot normalize. The results which follow will then be modified in a straightforward fashion.

[20] If the number of factors $\ell \to \infty$, but n grows sufficiently more rapidly, I would conjecture that the equilibrium weights would approach $\sigma_i^2/(\sigma_1^2+\ldots+\sigma_\ell^2)$.

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