Should the Two Parameter Capital Market Theories Be Extended To Higher Order Moments?

by

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I. Introduction

Since Markowitz developed the portfolio analysis model [22] and the Tobin-Sharpe-Lintner-Fama [26, 23, 17, 6] mean-variance capital market theory based on it evolved, there have been some suggestions that the third and possibly even higher order statistical moments should also be considered in the analysis [1, 2, 12, 13, 19, 20, 27]. These suggestions have been strengthened recently by empirical work and math derivations which have been published.

In particular, two recent studies motivated this inquiry into the effects of higher order statistical moments on investors' decisions.[1] First, Arditti published two empirical studies which both concluded that third moments tend to have significant inverse relations with the average asset's yield [1, 2]. Second, Jean derived some mathematical models which show how higher order moments might affect asset prices in equilibrium [13].

This paper analyzes the merit of these suggestions to consider higher order statistical moments. Then, empirical data for 119 mutual funds and 788 New York Stock Exchange (NYSE) stocks is examined: it shows how higher order moments have been historically related to average rates of return. The new data presented here suggests tentative conclusions about the effects of higher order moments and about those models which suggest their use.

II. Arditti's Data on Third Moments

Arditti has published two studies which both indicate that third statistical moments may have a significant inverse relation to the average returns investors demand to induce them to buy risky market assets. His first study used data for all the stocks in the Standard and Poor's Composite Index over the 1946-63 period [2]. Regressing the standard deviation (denoted σ_2) and skewness (denoted S) onto the geometric mean return[2] (denoted g) yielded Arditti's equation (2.1) below

$$g_i = .1044 + .4221\sigma_{2i} - .1677S_i \qquad \text{Arditti's eqn.(2.1)}$$
$$\qquad\qquad (5.0) \qquad (-3.8)$$

From equation (2.1) Arditti concluded that "the [significant] negative coefficient of the skewness variable means that the market likes positive skewness" [2, p. 256].

Arditti's second study was more forceful in asserting that the third moment was a primary factor in investor decisions [1]. This study used Sharpe's data for 34 mutual funds from the 1954-63 period [24]. Arditti's regression equation (1) is shown below.

$$\bar{r}_i = .0454 + .5760\sigma_{2i} - 7.059\sigma^3_{3i} \qquad R^2 = .7391$$
$$\qquad (.0697) \quad (-3.2102) \qquad \text{Arditti's eqn.(1)}$$

Arditti's equation (1) suggest that the variance has an <u>insignificant</u> effect on the average return demanded by investors. But, the third moment seems to weigh <u>heavily</u> in portfolio investment considerations.

Arditti's data may come as a surprise to some analysts who have been working primarily with the mean and variance models and had decided not to work with models involving higher order moments. Closer scrutiny of Arditti's work will lessen any such shock however. Arditti's results should be viewed with a bit of skepticism for several reasons. First of all, in the same study [2] in which he published his equation (2.1) shown above, Arditti published another regression which was similar to equation (2.1) except that it contained no industry dummy variable. This other regression yielded an insignificant regression coefficient for the skewness variable. The skewness effects shown in equation (2.1) are so weak they were washed out by industry effects.

A second reason Arditti's implications shouldn't be taken too literally is that the sample underlying his regression equation (1) is small and may be unrepresentative with respect to skewness. Monthly data for 32 of the 34 portfolios Arditti used to obtain his regression equation (1) above (plus data on many other funds too) was gathered independently at the Wharton School [11] for the period from Jan. 1960 to June 1968.[3] Since this sample data overlaps the data Arditti used for the 4 years from 1960 to 1963 inclusive, one would suspect that statistics for the 2 samples should be similar. However, such was not the case. Arditti reported [1, page 911] that the average skewness of his 34 mutual funds was positive. However, the Wharton School data for 32 of Arditti's 34 funds shows that all 32 had negative skewness.[4] In fact, the Wharton School data for 119 mutual funds from January 1960 to June 1968 used for this and other studies [11] has a total of only 4 mutual funds with positive skewness.

The point of all this is that Arditti's two studies and his regressions (2.1) and (1) above are not conclusive and possibly not even indicative of a significant market tendency.

III. Jean's Models

Jean derived math models which explicitly relate the market risk premiums [that is, the excess return over the riskless rate of interest] for portfolios and individual assets to the higher order moments of their probability distributions.

III A. Derivation of the Models

Jean's model is based on the same assumptions Sharpe and others have made [9, pp. 111-113]. If the first n moments of the probability distribution of returns are considered, the linear locus of portfolios (denotes e) which may be generated by borrowing or lending at the riskless rate R and placing any remaining funds in portfolio m is defined by the series of equations (1).

$$E(r_e) = f\,E(r_m) + (1-f)R \tag{1a}$$

$$\sigma_{2e} = f\,\sigma_{2m} \tag{1b}$$

$$\sigma_{3e} = f\,\sigma_{3m} \tag{1c}$$

$$\sigma_{4e} = f\,\sigma_{4m} \tag{1d}$$

$$\vdots$$

$$\sigma_{ne} = f\,\sigma_{nm} \tag{1n}$$

where f is the fraction of the equity of the portfolio made of assets R and m which is invested in m; $E(r_m)$ is the expected rate of return from the portfolio denoted by m; σ_{2m} is the standard deviation of the rates of return from portfolio m, and $E(r_i)$ and σ_{2i} are the expected return and standard deviation of the i^{th} efficient portfolio, respectively; and, σ_{ne} is the n^{th} root of the n^{th} statistical moment for portfolio e.

The slope of the linear opportunity locus for the portfolios denoted e in the direction of σ_n in n-dimensional moment space {that is, in $[E(r),\sigma_2,\sigma_3,\ldots\sigma_n]$ space} is shown in equation (2).

$$\partial[E(r) - R]/\partial\sigma_{ne} = \frac{E(r_m) - R}{\sigma_{nm}} \qquad (2)$$

Equation (2) may be derived by substituting equation (1n) into (1a), solving for the risk premium, and taking the partial derivative indicated.

Jean tried to derive the i^{th} individual assets' risk premium for the n^{th} moment -- when n is any positive integer. Using the same approach as Sharpe [23] and Fama [6], Jean worked with a hypothetical portfolio (which we will denote h) composed of the portfolio m and the i^{th} risky asset. He assumed that some fraction b of the equity of this portfolio is invested in the i^{th} asset and that the remainder (1-b), is invested in m.

As the fraction b is varied in the hypothetical portfolio denoted h, a locus of risk-return pairs -- that is, $\{\sigma_{nh}, E(r_h)\}$

pairs -- are traced out which are described by equation (3). Equation (3) is obtained by taking the indicated derivatives from the formula for the portfolio's expected return and the formula for the n^{th} root of the n^{th} statistical moment.[5]

$$\frac{dE(r_h)}{d\sigma_{nh}} = \frac{dE(r_h)}{db} \cdot \frac{db}{d\sigma_{nh}} \tag{3}$$

At b=0 portfolio m is attained -- that is, h = m at b = 0. Evaluating equation (3) at b=0 yields equation (4).

$$\left.\frac{dE(r_h)}{d\sigma_{nh}}\right|_{b=0} = \frac{(\sigma_{nm})^{n-1}\{E(r_i) - E(r_m)\}}{E(d_i d_m^{\,n-1}) - (\sigma_{nm})^n} \tag{4}$$

d_i is a random variable representing the deviation of the i^{th} assets return from its expected return, $d_i = [r_i - E(r_i)]$: d_m is analogously defined for portfolio m. Equating the marginal rate of transformation of risk for return which the investor can create for himself by varying b {that is, equation (4)} to the comparable rate of substitution available in the market. {that is, equation (2)} yields an equality which should prevail in equilibrium. This equilibrium condition may be solved to yield equation (5).

$$E(r_i) - R = \left[\frac{E(r_m) - R}{(\sigma_{nm})^n}\right] E(d_i d_m^{\,n-1}) \tag{5}$$

Equation (5) is the risk premium associated with the i^{th} asset's n^{th} statistical moment.[6] The right-hand side (rhs) of this equation is the product of two factors. The first factor on the rhs is the slope of efficient frontier in the direction of the n^{th} moment. The second factor on the right-hand side of equation (5) is the contribution of the i^{th} asset to the n^{th} moment of an efficient portfolio -- it measures the interaction of the i^{th} asset with portfolio m.

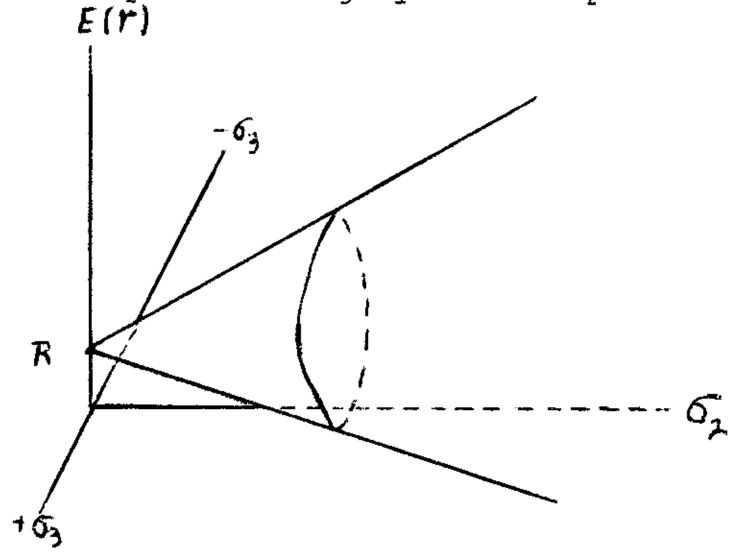
The seeming generality of Jean's findings are made apparent by noting that Sharpe's Capital Market Line [23,9] is merely a special case of equations (1a) and (1b) solved simultaneously for the portfolio's expected return. And, Sharpe's Security Market Line [9] is just a special case of equation (5) when n=2.

III B. Portfolio m Undefined in Jean's Model

Problems exist with the mathematical foundation of Jean's model because the desirable portfolio m is not a unique portfolio (like Fama's market portfolio [6]), nor is it even one particular portfolio. There are, in fact, numerous desirable portfolios which could be denoted m. For example, in the three dimensional $[\sigma_3, \sigma_2, E(r)]$ space shown in Figure 1 the opportunity set of all possible investments is shown as a cone with its vertex at the riskless rate R.

FIGURE I

The Opportunity Set of Investments in $[\sigma_3, \sigma_2, E(r)]$ Space Is Roughly Cone Shaped



The portfolios which will be most desirable to risk-averters are those which will maximize θ at each level of skewness, that is, those defined in equation (7).

$$\max \theta \Bigg|_{\sigma_3} = \frac{E(r_m) - R}{\sigma_{2m}} \Bigg|_{\sigma_3} \tag{7}$$

By varying σ_3 while maximizing θ the top half of the cone in Figure 1 will be traced out. This top half of the cone is the efficient frontier in $[\sigma_3, \sigma_2, E(r)]$ space. The main point, however, is that there will be a <u>different portfolio m</u> which has the maximum θ for each σ_3. Thus, there will be as many different portfolios denoted m as there are values of σ_3. Only in the special case when $\sigma_3 = 0$ for all assets will portfolio m be Fama's unique market portfolio.

Since portfolio m is not unique in Jean's model (nor is it even defined), then all the statistics and opportunity loci based on m are not defined either. For example, the contributions to the various moments, $E(d_i d_m^n)$, shown in equation (5) are not defined for models based on three or more moments. In fact, except for Sharpe's original two parameter models, none of Jean's asset pricing models are completely specified.

III C. Some of Investor's Preferences Vary With Sign of σ_{nm}

Another weakness of Jean's models is that it implies that investor's preferences with respect to the odd-numbered moments varies arbitrarily with the sign of the odd-numbered moments for the market portfolio; this is true for both the portfolio pricing model (2) and the asset pricing model (5). Consider, for example, the third moment. If σ_{3m} is positive, then equations (2) and (5) imply that investors prefer negatively skewed probability distributions to positively skewed distributions. This implication is difficult to take seriously. The sign of σ_{3m} should not have any dramatic impact on investor's preferences for other assets.

IV. Empirical Tests

IV A. Introduction to the Empirical Tests

In spite of some problems which have been pinpointed above, the mounting body of theoretical and empirical evidence points toward the need to consider more fully the possible effects of higher order moments on investor's decisions. Therefore, an analysis of empirical data was undertaken.

For the empirical tests reported here historical data for 788 individual common stocks listed on the New York Stock Exchange and 119 mutual funds were used.[7] Data for all 907 assets were gathered from an 8-1/2 year sample period. Thirty-four quarterly rates of return from January 1960 to June 1968 inclusive were calculated with dividends and cash disbursements reinvested and compounded.[8]

The n^{th} root of the n^{th} statistical moment was calculated according to equation (8) from historical frequency distributions of T=34 quarterly rates of return for the i^{th} asset.

$$\sigma_{ni} = \left[\left(\frac{1}{T}\right) \sum_{t=1}^{T} (r_{it} - \bar{r}_i)^n \right]^{1/n} \qquad (8)$$

$\bar{r}_i$ denotes the arithmetic average of the T returns for the i^{th} asset.

Jean's models suggest that the n^{th} root of the n^{th} statistical moment may have linear additive effects on the mean return (to a first approximation). Therefore, the n^{th} root of the n^{th} moment is used as the independent variable rather than the raw statistical moments or normalized statistical moments (namely, skewness and kurtosis).[9]

The contribution to the n^{th} moment risk factor [that is, the independent variable in Jean's equation(5)] was calculated with T=34 historical rates of return according to equation (9) below. Contribution to the n^{th} moment risk factor equals

$$E(d_i d_m^{n-1}) = \left[\left(\frac{1}{T}\right) \sum_{t=1}^{T} (r_{it} - \bar{r}_i)(r_{mt} - r_m)^{n-1} \right]. \qquad (9)$$

The Lorie-Fisher Combination Link Investment Relative with dividends reinvested [8] was used as a surrogate for the market portfolio, denoted m. Of course, as explained above, Jean's model presumes a seperate portfolio m for each level of skewness. However, for empirical tests it is quite expeditious to use only one portfolio m. Since the numerous portfolios defined by equation (7) which could be denoted portfolio m will all be highly positively correlated, and, the range of skewness they encompass is not very broad,[10] this simplification should not introduce such gross biases as to preclude its use in testing Jean's model (5).

Only the first four statistical moments were considered in this study. The meanings of the fifth and higher order moments are not generally understood. And, since there are no cogent reasons to suspect that the fifth and higher order moments contain valuable information, they were not examined.

The geometric mean return (g) defined in equation (10) was used as a surrogate for expected returns.[11]

$$g = \sqrt[34]{(1+r_1)(1+r_2)\ldots(1+r_{34})} - 1.0 \qquad (10)$$

IV B. <u>The Data For the Portfolio Model</u>

The hypothesis that a portfolio's expected rate of return or risk premium is a function of its second, third and fourth moments is represented symbolically by equation (11).

$$E(r_i) = f(\sigma^2_{2i}, \sigma^3_{3i}, \sigma^4_{4i}) \qquad (11)$$

The simplest and most plausible form which this function may be expected to assume is the linear additive form shown in equation (12b).[12]

$$E(r_i) = R + b_2\sigma_{2i} + b_3\sigma_{3i} + b_4\sigma_{4i} \tag{12b}$$

Table 1 shows the regressions, their t statistics and coefficients of determination ($\bar{R}^2$) for empirical estimates of equation (12b) using the sample of 119 mutual funds.[13]

Table 1

Regressions of Higher Order Statistical Moments Onto Their Mean Returns for 119 Mutual Funds

Regression No.	Empirical Estimates of Equation (12b)	$\bar{R}^2$
1	$gr_i = .00897 + \underset{(8.47)}{.21714}\sigma_{2i}$	.380
2	$gr_i = .00955 + \underset{(8.11)}{.24373}\sigma_{2i} + \underset{(1.66)}{.04032}\sigma_{3i}$	.395
3	$gr_i = .00928 + \underset{(3.21)}{.51032}\sigma_{2i} + \underset{(0.19)}{.00619}\sigma_{3i} - \underset{(-1.71)}{.20020}\sigma_{4i}$	.409

All t-values are in parentheses.

Unlike Arditti's data, the statistics in Table 1 suggest that only the first two moments have a significant relationship. The third moment is clearly insignificant for this sample.[14] The coefficient for the third moment also has the wrong sign. And, the fourth moment's regression coefficient is insignificant, but noteworthy nevertheless. This data implies that the mean-variance portfolio models do not waste significant information contained in higher order moments.[15]

IV C. Statistics for Jean's Contribution to Moments Model.

Jean's equation (5) hypothesizes a linear relationship between the assets' contributions to their higher order moments (for example, the covariance's contribution to the second moment) and their mean returns. Equation (13) depicts this model symbolically in a general form.

$$E(r_i) = h[E(d_i d_m),\ E(d_i d_m^2),\ E(d_i d_m^3)] \qquad (13)$$

The most obvious form[16] of equation (13) to test is the linear additive form shown as equation (14a).

$$E(r_i) = R + b_2 E(d_i d_m) + b_2 E(d_i d_m^2) + b_3 (d_i d_m^3) \qquad (14a)$$

Tables 2 and 3 show the statistics obtained by estimating equation (14a) with the data for the 119 mutual funds and the 788 NYSE stocks, respectively.

The data in Table II for the portfolios provides only limited support for Jean's contribution to higher order moments model (5). However, this lack of significant results is not conclusive in view of the simplification employed by using only one portfolio for portfolio m as explained above. The common stock data shown in Table III provides even less support for equation (5).

Considering the unsuspected positive signs on some of the regression coefficients for $E(d_i d_m^2)$; the different signs on the

Table II

Regression of Mean Returns Onto Higher Order Moment Risk Contribution Factors for 119 Mutual Funds

Regression No.	Empirical Estimates of Equation (5)	$\bar{R}^2$
4	$g_i = .00781 + \underset{(9.63)}{2.96082}\, E(d_i d_m)$	.44
5	$g_i = .00806 + \underset{(10.28)}{4.1626}\, E(d_i d_m) + \underset{(4.20)}{22.39907}\, E(d_i d_m^2)$	.516
6	$g_i = .00829 + \underset{(4.31)}{7.84999}\, E(d_i d_m) - \underset{(-.09)}{8.36678}\, E(d_i d_m^2) -$	
	$\underset{(-2.07)}{127.02388}\, E(d_i d_m^3)$	.53

Table III

Regressions of Mean Returns Onto Higher Order Moment Risk Contribution Factors for 788 Individual Common Stocks

Regression No.	Empirical Estimates of Equation (5)	$\bar{R}^2$
7	$g_i = .01932 + \underset{(4.81)}{1.09841}\, E(d_i d_m)$	.028
8	$g_i = .01949 + \underset{(4.75)}{1.09313}\, E(d_i d_m) + \underset{(0.196)}{.46867}\, E(d_i d_m^2)$	.028
9	$g_i = .01961 + \underset{(2.02)}{1.27447}\, E(d_i d_m) + \underset{(0.10)}{0.25378}\, E(d_i d_m^2)$	
	$- \underset{(-0.31)}{6.01987}\, E(d_i d_m^3)$	.028

$E(d_i d_m^3)$ coefficient in regressions 6 and 9; and, the weak results shown in Table III,[17] only the two parameter models previously suggested by Sharpe [23] seem to enjoy statistical support.

Conclusions

The suggestions that more than just the first two moments should be considered in asset pricing models seem to lack strong empirical grounds or a rigorous model to support them. Arditti's regression results are suspect. And, Jean's asset pricing models based on several higher order moments are not fully developed. Furthermore, the empirical tests presented in Tables I, II and III did not attest to the significance of the third and fourth statistical moments. The two parameter models developed by Sharpe and others appear to be the most well-developed and the most well supported by empirical data of the models existing at the present.

FOOTNOTES

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[1]In addition to these more recent suggestions mentioned above, there have been other studies which also implied that more than only the mean and variance of assets' probability distributions of returns should be considered [10,21].

The classic Friedman-Savage utility study suggests a fourth order utility function [10]. The expected utility of this fourth degree utility of wealth function can be shown to be a function of the first four moments about the origin (which in turn determine the first four statistical moments) of the probability distribution of terminal wealth or rates of return. (See footnote 12).

The stochastic dominance literature does not suggest the use of higher order moments per se [3, 12, 19, 20, 27]. Instead it shows that basing investment decisions on only the mean and variance can sometimes lead to the selection of efficient assets which are less desirable than some inefficient assets. This proves that some useful information about the probability distribution is wasted if only the mean and variance are considered. Examining higher order moments may give some insight into the question of how much valuable information is contained in these more esoteric aspects of the probability distribution.

[2]Regressing higher order moments onto the geometric mean return rather than the arithmetic mean return results in lower coefficients of determination. However, it has been this writer's experience that conclusions based on the t statistics of regressions which differ only in their use of the geometric and arithmetic means as the independent variable usually do not differ. Therefore, Arditti's regression will be compared to the regressions presented later in this paper which use the arithmetic mean return as the dependent variable.

[3]It is possible that the credit crisis of 1966 which caused stock prices to drop sharply could down bias the skewness statistics used here.

[4]The rate of return measurement used in empirical work can influence the skewness of the probability distribution. If returns are defined to be the price plus dividend relatives minus unity, $r = \frac{P_{t+1} + D}{P_t} - 1$, instead of the log of the relative, $\ln\left(\frac{P_{t+1} + D}{P_t}\right)$,

the distribution will be more positively skewed. Since the first definition was used in this study the skewness estimates should not be low.

[5] The n^{th} root of the n^{th} moment of a hypothetical portfolio is defined below.

$$\sigma_{nh} = \left[E(bd_i + (1-b)d_m)^n\right]^{1/n}$$

$$= \left[\sum_{k=0}^{n} \left(\frac{n!}{(n-k)!k!}\right. b^{n-k} (1-b)^k E(d_i^{n-k} d_m^k)\right]^{1/n}$$

The reciprocal of the derivative $\frac{d\sigma_{nh}}{db}$ is:

$$\frac{db}{d\sigma_{nh}} = \frac{\sigma^{n-1}}{E(d_i d_m^{n-1}) - (\sigma_{nm})^n}$$

The derivative $dE(r_n)/db = E(r_i) - E(r_m)$ is multiplied times $\frac{db}{d\sigma_{nh}}$ using the chain rule as shown in equation (3) to obtain equation (4).

[6] Jean's derivation of equation (5) was conducted in $[E(r), \sigma_n]$ space. The brief derivation outlined in the preceding footnote is different from Jean's original derivation only to the extent that it is done in $[\sigma_n, E(r)]$ space rather than $[E(r), \sigma_n]$ space.

Some analysts have erroneously interpreted equation (5) as implying an unsuspected inverse relation between the risk premium and the third moment. This is not true if σ_{3m} is negative, as the empirical data used here [11] indicates.

[7] The data for the 788 stocks was obtained from the University of Chicago Price Relative File, a magnetic tape described by L. Fisher and J.H. Lorie [8].

For each common stock in the sample, the equation below shows how the returns were calculated.

$$r_t = \frac{(P_t - P_{t-1}) + D_t}{P_{t-1}} = \frac{\text{Capital gains or losses plus dividends}}{\text{purchase price}}$$

P_t denotes the market price at the beginning of the month, D_t was the cash dividend (if any) paid during the period. All market prices were adjusted for changes in the unit of account (such as stock dividends and splits) before rates of return were calculated. Rates of return were calculated for the mutual funds as shown below.

[8]The raw mutual fund data was gathered for the Twentieth Century Fund Study of institutional investors [11].

The mutual fund relatives contained income from cash dividend disbursements (D_t), changes in net asset value (ΔV_t) plus any capital gains disbursements (g_t) made by the fund.

$$r_t = \frac{D_t + \Delta V + g_t}{\text{Beginning of period net asset value}}$$

The data files [8], [11] contained monthly relatives (mr) which were converted to quarterly relatives (qr) as follows:

$$qr = (mr_1)(mr_2)(mr_3) = \left(\frac{P_1}{P_o}\right)\left(\frac{P_2}{P_1}\right)\left(\frac{P_3+D}{P_2}\right) = \left(\frac{P_3+D}{P_o}\right)$$

Quarterly returns were used in preference to monthly returns because the short-run (that is, the monthly) data contains more random noise [5] than data measured over longer differencing intervals (namely quarters).

[9]Using the normalized statistical moments as the independent variables offers the advantage of increased statistical efficiency compared to using the raw statistical moments. However, taking the root of the statistical moments also dampens the tendency for the raw statistical moments to explode when outlying observations enter the calculations. And, since Jean's work suggests linear additive models when using the roots of the moments, the roots were used here. The proper adjustments to maintain the sign of the 3rd root of the third moment were made after taking the roots. Regressions were run using the statistical moments as the independent variables; then, the regressions were replicated using the roots of the moments. The roots of the moments yielded insignificantly better fits. So, the roots were used as the independent variables here. Problems with normalized moments (namely, kurtosis) also discouraged their use [15].

[10] Blume also reports no appreciable degree of skewness in an independent study [4, p.164].

[11] Empirical tests of <u>ex ante</u> theories which use <u>ex post</u> data must assume, either implicitly or explicitly, that investors past expectations were unbiased in order to justify using historical data. Since the geometric mean return is the true compound rate of return the investor earned from a multi-period (namely, 34 quarter) investment, it is used as the surrogate for expected returns so as not to violate the underlying assumption that investor's past expectations tended to be borne out in an unbiased manner.

[12] Taking a total differential of equation (11) yields equation (12)

$$d\{E(r_i)-R\} = \left(\frac{\partial[E(r_i)-R]}{\partial\sigma^2_{2i}}\right) d\sigma^2_{2i} + \left(\frac{\partial[E(r_i)-R]}{\partial\sigma^3_{3i}}\right) d\sigma^3_{3i} + \left(\frac{\partial[E(r_i)-R}{\partial\sigma^4_{4i}}\right) d\sigma^4_{4i} \quad (12)$$

$$d\{E(r_i)\} = R + \sum_{n=2}^{4} \left(\frac{\partial[E(r_i)-R]}{\partial\sigma^n_{ni}}\right) d\sigma^n_{ni} \quad (12a)$$

Since the statistical moments are all mathematical functions of (that is, dependent on) the lower order moments about the origin as shown below. The total differential of equation (12) is only an approximation because the moments are not independent.

$$\sigma^2_2 = E(r) - E(r)^2$$

$$\sigma^3_3 = E(r^3) - 3E(r)\,E(r^2) + 2E(r)^3$$

$$\sigma^4_4 = E(r^4) - 4\,E(r)\,E(r^3) + 6E(r)^2\,E(r^2) - 3E(r)^4$$

Equation (12) is only shown to suggest the rationale behind the linear additive regression model shown as equation (12b).

[13] Evidence has been published suggesting that the probability

distribution of returns is non-normal [5,7,18]. This implies that the t statistic tests of significance [14] underlying the discussion of Tables I, II, and III are biased. No cognizance is taken of such possible biases.

[14] It has been shown [16] that when the probability distribution is not symmetric, a spurious covariance is induced between the first two moments as follows: $\mathrm{Cov}(E(r_i),\sigma_i) \cong \sigma_3^3/n$, where n is the sample size. However, the degree of skewness found in the data presented in Tables I, II and III is small enough so that this bias is not highly significant.

[15] Fama's empirical research on security price movements [5] suggests that the probability distributions of stock prices cannot be well-described by statistical moments because the data implies a theoretical probability distribution which has an infinite variance. Fama suggests that a four parameter Paretian distribution which can be described by a location parameter, a characteristic exponent, a skewness parameter, and a scale parameter can best describe the probability distributions of stocks. Sidestepping the question of which probability distribution is the best description of reality, the fact remains that any empirical sample of stock prices will in fact have a finite variance. This is obvious since economic data are finite real numbers. Therefore (whether or not they are efficient statistics) the first four moments of a stock's probability distribution exist and may provide valuable information. Since the moments exist (in the empirical sense that there are finite numbers), they were used in this research for two main reasons:

1) Arditti's and Jean's work was stated in terms of statistical moments, and
2) for statistical work the moments are simpler to work with than the parameters of the Paretian distribution.

[16] Taking a total differential of equation (13) suggests, as a first approximation (that is, overlooking dependence between the variables), equation (14) as an asset pricing model.

$$d[E(r_i)-R] = \sum_{n=1}^{3} \left(\frac{\partial E(r_i)}{\partial E(d_i d_m^n)} \right) dE(d_i d_m^n) \qquad (14)$$

Equation (14) implies equation (14a).

[17] Theory and empirical evidence suggest that the sign on the $E(d_i d_m)$ term will be positive in bull market periods and negative in bear market periods. Therefore, the conclusion that $E(d_i d_m)$ is significant is based on the absolute value of its t statistic rather than on its sign or actual value.

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